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Laser pulse self-compression in an active fibre with a finite gain bandwidth under conditions of a nonstationary nonlinear response

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Abstract. **We study the influence of a nonstationary nonlinear response of a medium on self-compression of soliton-like laser pulses in active fibres with a finite gain bandwidth. Based on the variational approach, we qualitatively analyse the self-action of the wave packet in the system under consideration in order to classify the main evolution regimes and to determine the minimum achievable laser pulse duration during self-compression. The existence of stable soliton-type structures is shown in the framework of the parabolic approximation of the gain profile (in the approximation of the Gnizburg–Landau equation). An analysis of the self-action of laser pulses in the framework of the nonlinear Schrödinger equation with a sign-constant gain profile demonstrate a qualitative change in the dynamics of the wave field in the case of a nonstationary nonlinear response that shifts the laser pulse spectrum from the amplification region and stops the pulse compression. Expressions for a minimum duration of a soliton-like laser pulse are obtained as a function of the problem parameters, which are in good agreement with the results of numerical simulation.**

Keywords: *self-compression of laser pulses, active fibre, nonstationary nonlinear response.*

1. Introduction

The issue of laser generation of high-energy ultrashort laser pulses is an area of active experimental research and an object of theoretical physics that studies the nonlinear dynamics of wave fields with time scales comparable with a field cycle. The importance of this range of tasks is due to a large number of their applications in science, engineering, and technology. As such, we note the study of ultrafast processes, generation and detection of THz radiation [1, 2], generation of high harmonics [3, 4], formation of attosecond pulses [5], and acceleration of electrons and ions [6].

Currently, there exist several approaches to the generation of ultrashort laser pulses. The most attractive one relies on the use of solid-state laser systems based on broadband amplification, for example, in Ti:sapphire crystals and/or in parametric amplifiers. They can enable the generation of ultrashort laser pulses with sufficiently high energies [7, 8]. One more technique of obtaining high-power laser pulses is based on the broadening of a wave-packet spectrum in a high-pressure gas and on the subsequent compression of a wave packet using gratings or chirped mirrors. An alternative method is self-compression of laser pulses (a decrease in the duration of a wave packet without the use of external linear dispersion elements) during their propagation in a medium with Kerr [9, 10] or ionisation [11–13] nonlinearities. A similar mechanism for generating petawatt laser pulses uses self-compression due to the relativistic nonlinearity of plasma [14, 15].

At the same time, it is necessary to single out laser pulse shortening mechanisms, which are associated with an adiabatic change in the parameters of a soliton-type wave field during its propagation in a nonlinear medium. In this case, one can expect to obtain wave packets having a good temporal contrast. We note some papers in which self-compression of soliton-like laser pulses was experimentally and theoretically investigated: self-compression in a gain medium [16, 17], in a waveguide system with a monotonically decreasing linear dispersion [18, 19], and in a waveguide in the presence of cubic and ionisation nonlinearities [20], as well as adiabatic self-compression of three-dimensional laser pulses during their self-focusing when the dispersion length is less than the diffraction length [21–23].

Thus, the successful development of fibre lasers stimulates the study of the possibility of replacing components of solid-state laser systems with fibre-optic ones, which can dramatically increase the attractiveness of relevant applications. Being inferior to solid-state systems in power characteristics, fibre lasers and nonlinear optical devices have such advantages as high efficiency of conversion of pump energy into radiation energy caused by the waveguide geometry, efficient heat removal and high quality of the laser beam spatial profile, as well as low cost, compactness, and lack of adjustment during operation.

Recently, the idea of amplifying wave packets in an array of independent waveguides has been discussed to obtain laser pulses with extremely high power [24, 25]. In a medium with an anomalous group velocity dispersion, amplification of laser pulses can also be accompanied by an adiabatic decrease in the laser-pulse duration [16, 17] if a soliton-like laser pulse, whose dispersion length is much smaller than the inverse gain increment, is injected at the fibre input. In this case, amplification of the wave field can be regarded as a small perturbing factor during the wave-packet propagation. The requirement that the gain increment be small increases the need for using significant lengths of interaction between the radiation and the medium. Obviously, in the course of the adiabatic decrease

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in the laser-pulse duration, there can arise various negative factors such as nonlinear response nonstationarity, higher-order dispersion, and possible inhomogeneities in fibre.

Peculiarities of self-action of laser pulses are mainly theoretically analysed within the framework of the nonlinear Schrödinger equation (NLSE) and its generalisations. In the NLSE, allowance for additive dissipation, amplification, and other effects determined by the physical problem leads to another well-known Ginzburg–Landau equation [26–29]. The localised structures that arise here are called dissipative solitons [26, 29]. In contrast to the one-parameter family of solitons, in the NLSE the competition of dissipation and amplification in the nonconservative regime makes the situation more definite and stable. Owing to the complexity and variety of the processes under consideration, a theoretical study in many papers reduces to a numerical modelling of the problem, since the nonconservatism of the system significantly complicates the use of qualitative methods for studying the evolution of wave fields.

The purpose of this work is to investigate the influence of the Raman nonlinearity on self-compression of soliton-like laser pulses in active fibres with a finite gain bandwidth using an analytical apparatus of nonlinear optics. Using the variational approach, we qualitatively analyse the self-action of the wave packet in the system under consideration in order to classify the main regimes of the wave-field evolution and to determine the minimum achievable laser pulse duration during self-compression as a function of the problem parameters.

The paper has the following structure. In Section 2, a basic equation is formulated for investigating the self-action of a laser pulse in an active fibre with a finite gain bandwidth, and the variational approach to the nonconservative case is generalised. Section 3 presents an analytical and numerical analysis of the nonlinear dynamics of a wave packet on the basis of the Ginzburg–Landau equation. For the analytical investigation of processes in the system, a closed system of ordinary differential equations for the characteristic parameters of a laser pulse having a Gaussian intensity distribution is obtained using the variational approach. The existence of stable soliton-type structures is shown that are 'pressed' in the spectral region to the edge of the gain band of the active medium. The minimum duration of the soliton and its frequency shift due to nonstationarity of the nonlinear response are determined as functions of the problem parameters. Section 4 theoretically analyses self-compression of laser pulses within the framework of the nonlinear Schrödinger equation under conditions of a constant-sign gain profile. Using the variational approach, a closed system of equations describing the dynamics of the laser pulse parameters and an expression for the minimum duration of a soliton-like laser pulse are obtained as functions of the problem parameters. In Conclusions we summarise the results of the work.

2. Formulation of the problem

Let us consider the self-action of subpicosecond laser pulses in an active optical fibre with a finite gain bandwidth, taking into account the instantaneous electronic (Kerr) and delayed molecular (Raman) nonlinearities. We assume that the carrier frequency of the laser pulse lies in the region of the anomalous group-velocity dispersion, and the fibre radius is small (on the order of the wavelength), so that the propagation in the fibre is single-mode. The latter requirement automatically means that we neglect the possibility of the wave-field self-focusing inside the fibre. It should be noted that the gain saturation of the wave packet can be ignored, because the fibre laser media have a high saturation energy. For example, the saturation energy of erbium laser fibres is $\sim$10 μJ, and the energy of injected short laser pulses is less than 1 nJ. As a result, the evolution of laser pulses in this case can be described with the help of the NLSE:

$$\mathrm{i}\frac{\partial E}{\partial z} + \mathrm{i}\frac{\partial k}{\partial \omega}\frac{\partial E}{\partial t} - \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\frac{\partial^2 E}{\partial t^2} + \frac{n_2\omega_0}{c}|E|^2 E$$
$$= \tau_{\mathrm{R}}\frac{n_2\omega_0}{c}E\frac{\partial |E|^2}{\partial t} + \frac{\mathrm{i}\omega_0}{4\pi n_0 c}\int_{-\infty}^{+\infty}\chi(\omega)E(z,\omega)\exp(-\mathrm{i}\omega t)\mathrm{d}\omega.$$

Here, ω_0 is the carrier frequency of the laser pulse; n_0 and n_2 are linear and nonlinear refractive indices; $\chi(\omega)$ is the imaginary part of the medium susceptibility; and R is the characteristic time for establishing the nonlinearity. In the accompanying coordinate system moving with the group velocity of the wave packet, the NLSE can be written in dimensionless variables [27]:

$$\mathrm{i}\frac{\partial \Psi}{\partial z} + \frac{\partial^2 \Psi}{\partial \tau^2} + |\Psi|^2\Psi - \mu\Psi\frac{\partial |\Psi|^2}{\partial \tau}$$
$$- \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}G(\omega)\tilde{\Psi}(z,\omega)\exp(-\mathrm{i}\omega\tau)\mathrm{d}\omega = 0. \quad (1)$$

Here, the longitudinal coordinate $\tau = t - (\partial k/\partial\omega)z$ and the evolutionary coordinate z are normalised to the characteristic laser pulse duration τ_{eff} and the corresponding dispersion length $z_0 = 2\tau_{\mathrm{eff}}^2/(\partial^2 k/\partial\omega^2)$; $\Psi = E\sqrt{n_2\omega_0 z_0/c}$ is the complex amplitude of the wave packet envelope;

$$\tilde{\Psi}(z,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Psi(z,\tau)\exp(\mathrm{i}\omega\tau)\mathrm{d}\tau;$$

$\mu = \tau_{\mathrm{R}}/\tau_{\mathrm{eff}}$; and $G(\omega) = \omega_0 z_0\chi(\omega)/(2n_0 c)$. The second term in (1) describes the linear dispersion of the medium, the third term is responsible for the Kerr nonlinearity, and the fourth term is responsible for the nonlinear response inertia. The last term describing the process of laser pulse amplification in an active medium is represented through the spectral gain $G(\omega)$.

An obvious simplification of the original equation (1) is an analysis of the self-action of a quasi-monochromatic wave packet with a spectral width $\Delta\omega$, injected into the active fibre with a large gain bandwidth Ω: $\Omega > \Delta\omega$. The expansion of the gain $G(\omega)$ in a Taylor series near the carrier frequency to the quadratic term $G(\omega) \approx \gamma - D\omega^2$ allows us to pass from equation (1) to the well-known Ginzburg–Landau equation [26, 27, 29]:

$$\mathrm{i}\frac{\partial \Psi}{\partial z} + \frac{\partial^2 \Psi}{\partial \tau^2} + |\Psi|^2\Psi = \mu\Psi\frac{\partial |\Psi|^2}{\partial \tau} + \mathrm{i}\gamma\Psi + \mathrm{i}D\frac{\partial^2 \Psi}{\partial \tau^2},$$
$$D = \frac{\gamma}{\Omega^2}. \quad (2)$$

Equation (2) in the absence of Raman nonlinearity ($\mu = 0$) has been well studied in a number of papers [27, 29, 30]. Note that in the framework of equation (2), in the case $\mu = 0$, there exist stable soliton solutions, which are called dissipative solitons [26].

Further analytical and numerical analysis of the influence of the Raman nonlinearity ($\mu \neq 0$) on the mechanism of adiabatic reduction of the laser-pulse duration during its amplification in the active medium will be performed on the basis of equations (1) and (2). To obtain analytical estimates, we use the variational approach. This will allow us to obtain a closed system of ordinary differential equations for the characteristic integral parameters of the distribution of the Gaussian wave field (laser-pulse energy, duration, and chirp). Despite the complication of the situation in the active medium due to the absence of a Hamiltonian, the variational problem can be formulated both for the Ginzburg–Landau equation (2) and for the initial equation (1). Moreover, along with the Lagrangian of the conservative part of the system

$$\mathcal{L} = \int_{-\infty}^{+\infty}\left[\frac{\mathrm{i}}{2}\left(\Psi\frac{\partial\Psi^*}{\partial z} - \Psi^*\frac{\partial\Psi}{\partial z}\right) - \frac{1}{2}|\Psi|^4 + \left|\frac{\partial\Psi}{\partial\tau}\right|^2\right]\mathrm{d}\tau, \quad (3)$$

which coincides for equations (1) and (2), it is necessary to determine the dissipative function of the system δQ.

We confine ourselves to the evolution of Gaussian wave packets:

$$\Psi = \sqrt{\frac{W}{\sqrt{\pi}\tau_\mathrm{p}}}\exp\left[-\frac{(\tau-q)^2}{2\tau_\mathrm{p}^2} + \mathrm{i}\beta(\tau-q)^2 - \mathrm{i}\bar{\omega}(\tau-q) + \mathrm{i}\theta\right]. \quad (4)$$

Here,

$$W = \int_{-\infty}^{+\infty}|\Psi|^2\mathrm{d}\tau$$

is the laser pulse energy; and τ_p, β, $\bar{\omega}$ and θ are, respectively, the pulse duration, chirp, frequency and phase of the wave field at the intensity maximum of the packet, whose position is determined by the parameter $q(z)$. In the conservative case, the change in the parameters $a_j = \{W, \tau_\mathrm{p}, q, \beta, \bar{\omega}, \theta\}$ during the pulse propagation is described by the Euler equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial\bar{\mathcal{L}}}{\partial\dot{a}_j} - \frac{\partial\bar{\mathcal{L}}}{\partial a_j} = 0 \quad \left(\dot{a}_j = \frac{\mathrm{d}a_j}{\mathrm{d}z}\right) \quad (5)$$

using the truncated Lagrangian $\bar{\mathcal{L}}$ – the Lagrange function (3) – calculated on a given field distribution (4):

$$\bar{\mathcal{L}} = \frac{W\tau_\mathrm{p}^2}{2}\frac{\mathrm{d}\beta}{\mathrm{d}z} + W\frac{\mathrm{d}\theta}{\mathrm{d}z} - W\bar{\omega}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{W}{2}\left(\frac{1}{\tau_\mathrm{p}^2} + 4\beta^2\tau_\mathrm{p}^2\right) + \bar{\omega}^2 W - \frac{W^2}{\sqrt{8\pi}\tau_\mathrm{p}}. \quad (6)$$

A generalisation of the Euler equation for determining the parameters of the variable function (4) in the nonconservative case consists in taking into account the contribution of the dissipative part of the system [29]:

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial\bar{\mathcal{L}}}{\partial\dot{a}_j} - \frac{\partial\bar{\mathcal{L}}}{\partial a_j} = \int\left[\frac{\delta Q}{\delta\Psi}\frac{\partial\Psi}{\partial a_j} + \mathrm{c.c.}\right]\mathrm{d}\tau. \quad (7)$$

3. Self-compression of laser pulses in the framework of the Ginzburg–Landau equation

Let us first analyse the self-action of a laser pulse in an active fibre with a finite gain bandwidth on the basis of the Ginzburg–Landau equation, taking into account the nonstationarity of the nonlinear response (2). For it, the variation of the dissipative function is

$$\delta Q = \int_{-\infty}^{+\infty}\left[\mathrm{i}\gamma(\Psi\delta\Psi^* - \Psi^*\delta\Psi) + \mathrm{i}D\left(\frac{\partial^2\Psi}{\partial\tau^2}\delta\Psi^* - \frac{\partial^2\Psi^*}{\partial\tau^2}\delta\Psi\right) + \mu\frac{\partial|\Psi|^2}{\partial\tau}\delta|\Psi|^2\right]\mathrm{d}\tau. \quad (8)$$

Accordingly, equations (7) for the parameters of the wave packet (4) in the considered nonconservative case have the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial\bar{\mathcal{L}}}{\partial\dot{a}_j} - \frac{\partial\bar{\mathcal{L}}}{\partial a_j} = \mu\int_{-\infty}^{+\infty}\frac{\partial|\Psi|^2}{\partial\tau}\frac{\partial|\Psi|^2}{\partial a_j}\mathrm{d}\tau + \mathrm{i}\gamma\int_{-\infty}^{+\infty}\left(\Psi\frac{\partial\Psi^*}{\partial a_j} - \mathrm{c.c.}\right)\mathrm{d}\tau + \mathrm{i}D\int_{-\infty}^{+\infty}\left(\frac{\partial^2\Psi}{\partial\tau^2}\frac{\partial\Psi^*}{\partial a_j} - \mathrm{c.c.}\right)\mathrm{d}\tau. \quad (9)$$

The right-hand side in (9) determines an additional contribution to the change in the longitudinal structure of the pulse associated with the dissipative factors in the original equation (2). Calculating and differentiating the integrals, we arrive at the system of ordinary differential equations for the parameters of the localised structure (4):

$$\frac{\mathrm{d}W}{\mathrm{d}z} = 2\gamma W - \frac{DW}{\tau_\mathrm{p}^2}(1 + 4\beta^2\tau_\mathrm{p}^4 + 2\bar{\omega}^2\tau_\mathrm{p}^2), \quad (10a)$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}z} = \frac{1}{\tau_\mathrm{p}^4} - 4\beta^2 - \frac{W}{\sqrt{8\pi}\tau_\mathrm{p}^3} - \frac{4D\beta}{\tau_\mathrm{p}^2}, \quad (10b)$$

$$\frac{\mathrm{d}\tau_\mathrm{p}}{\mathrm{d}z} = 4\beta\tau_\mathrm{p} + \frac{D}{\tau_\mathrm{p}} - 4D\beta^2\tau_\mathrm{p}^3, \quad (10c)$$

$$\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}z} = -\frac{\mu W}{\sqrt{2\pi}\tau_\mathrm{p}^3} - \frac{2D\bar{\omega}}{\tau_\mathrm{p}^2} - 8D\bar{\omega}\beta^2\tau_\mathrm{p}^2, \quad (10d)$$

$$\frac{\mathrm{d}q}{\mathrm{d}z} = 4D\beta\bar{\omega}\tau_\mathrm{p}^2 - 2\bar{\omega}. \quad (10e)$$

Note that equation (10e) for the position of centre of intensity of the wave packet is isolated from the rest of system (10). As a result, the problem of investigating the self-action of a laser pulse has been reduced to an analysis of the first four equations (10).

Let us first analyse the case of the absence of the Raman nonlinearity ($\mu = 0$). An analysis of the system of equations (10b)–(10d) shows the existence of a stable equilibrium point (knot which becomes the focus for $\mu \neq 0$) at $\bar{\omega} = 0$ and

$$\tau_\mathrm{p} = \tau_\mathrm{f} \equiv \frac{\sqrt{32\pi}}{W}\frac{1+D^2}{D^2}(\sqrt{1+D^2} - 1) \approx \frac{\sqrt{8\pi}}{W} \quad (11)$$

with a characteristic oscillation frequency $\tilde{\omega} \approx 2W^2/(4\pi) \gg \gamma$. Since the characteristic time of the change in energy $1/\gamma$ is much larger than the period of oscillations, the position of the centre (τ_f) and the amplitude ($\delta\tau_\mathrm{p} \propto 8\pi/W^2$) of the oscillations will vary adiabatically slowly with increasing energy W. This behaviour corresponds to the evolution of soliton-type field distributions, arising at a competition of anomalous group-velocity dispersion and cubic nonlinearity. Being stable particle-like formations, solitons maintain the relation of their duration and energy (amplitude) of form (11). In this case, the solutions for $D \ll 1$ have a weak frequency modulation:

$$\beta(z) \approx -\frac{D}{4\tau_{\rm p}^2(z)}. \tag{12}$$

This means that the structures differ from the NLSE solitons by the presence of a negative chirp. The presence of the chirp (12) practically did not change the ratio well known for NLSE solitons: $\tau_{\rm p} \propto 1/W$.

Near the focus, the quantity $\beta^2\tau_{\rm p}^4 \lesssim D^2/4 \lll 1$ for $D \ll 1$. Let us consider the most interesting case, i.e. $\bar\omega = 0$, which is an exact solution of equation (10d) for $\mu = 0$. This allows us to use relation (11) and equation (10a) to obtain an equation for the duration of a soliton-like pulse as it propagates in the fibre:

$$\tau_{\rm p}\frac{{\rm d}\tau_{\rm p}}{{\rm d}z} = D - 2\gamma\tau_{\rm p}^2. \tag{13}$$

The solution of this equation

$$\tau_{\rm p} = \sqrt{\frac{D}{2\gamma} + \left(1 - \frac{D}{2\gamma}\right)\exp(-4\gamma z)} \tag{14}$$

shows that the laser pulse duration decreases exponentially to the maximum possible pulse duration

$$\tau_{\rm lim} = \sqrt{\frac{D}{2\gamma}}. \tag{15}$$

We rewrite this expression by substituting $D = \gamma/\Omega^2$ into it:

$$\tau_{\rm lim} = \frac{1}{\sqrt{2}\,\Omega}. \tag{16}$$

It follows from (16) that the minimum possible duration of a soliton-like wave packet in the process of an adiabatic increase in the wave field energy is determined only by the gain bandwidth Ω.

Next, we take into account the influence of the nonlinear response nonstationarity of the medium ($\mu \neq 0$) on the self-action dynamics of the laser pulse in the active fibre. Since the carrier frequency $\bar\omega$ does not enter into equations (10b) and (10c), the equilibrium point [see (11) and (12)] of these equations is the same as in the case $\mu = 0$. This equilibrium point corresponds to the soliton-like distribution of the wave field. From relation (11) and equations (10a) and (10c) we obtain the equations determining a decrease in the duration $\tau_{\rm p}$ and the transformation of the carrier frequency $\bar\omega$ of the wave packet as it propagates in the fibre:

$$\frac{{\rm d}\tau_{\rm p}}{{\rm d}z} = -2\gamma\tau_{\rm p} + \frac{D}{\tau_{\rm p}}(1 + 2\bar\omega^2\tau_{\rm p}^2), \tag{17a}$$

$$\frac{{\rm d}\bar\omega}{{\rm d}z} = -\frac{2\mu}{\tau_{\rm p}^4} - \frac{2D\bar\omega}{\tau_{\rm p}^2}. \tag{17b}$$

It follows from Eqn (17b) that the Raman nonlinearity only leads to a down-shift of the soliton frequency (toward larger wavelengths). The rate of the frequency shift increases with decreasing soliton duration: ${\rm d}\bar\omega/{\rm d}z \propto -\mu/\tau_{\rm p}^4$ [31].

The above considered limiting case $\mu = 0$ is a special case of equations (17). A typical phase plane for the system of equations (17) in this limit is shown in Fig. 1a. It is seen that the duration of a soliton-like wave packet is bounded below by the value $\tau_{\rm lim}$ (16). In this case, the carrier frequency $\bar\omega$ tends to zero if its initial value is different from the carrier frequency of the gain band. The reason is that the gain in the spectral region of $G(\omega)$ reaches its maximum value only at $\bar\omega = 0$.

Analysis of the system of equations (17) in the general case ($\mu \neq 0$) shows the presence of an equilibrium state:

$$\tau_{\rm p} = \sqrt{-\frac{\mu}{D\bar\omega}} \equiv \sqrt{-\frac{\mu\Omega^2}{\gamma\bar\omega}}, \tag{18a}$$

$$\bar\omega = \frac{D^2 - \sqrt{D^4 + 16\gamma\mu^2 D}}{4D\mu} \equiv \frac{\gamma - \sqrt{16\Omega^6\mu^2 + \gamma^2}}{4\Omega^2\mu}, \tag{18b}$$

whose type at $\mu = 0$ corresponds to a node, and at $\mu \neq 0$ – to a focus. For the small coefficients $\mu \ll [D^3/(16\gamma)]^{1/2} = \gamma/(4\Omega^3)$, formula (18a) gives values close to $\tau_{\rm lim}$ found in (15):

$$\tau_{\rm lim} \approx \frac{1}{\sqrt{2}\,\Omega} + \frac{\sqrt{2}}{\Omega}\left(\frac{\mu\Omega^3}{\gamma}\right)^2, \quad \bar\omega_{\rm lim} \approx -\frac{2\mu\Omega^3}{\gamma}\Omega. \tag{19}$$

In this case, the minimum duration is determined primarily by the gain bandwidth of the active medium, Ω. At the same time, the maximum frequency shift $\bar\omega_{\rm lim}$ increases with increasing μ according to the linear law. Thus, the Raman nonlinearity leads only to a shift of the carrier frequency of the soliton to the long-wavelength part of the spectrum. As follows from formula (19), the frequency shift $\bar\omega$ can be reduced by increasing the gain of the medium, γ.

As an example, Fig. 1b shows the phase plane $\tau_{\rm p}, \bar\omega$ for the system of equations (17) at $\mu = 10^{-4}$, $\gamma = 0.3$, and $\Omega = 10$. We see that at the initial stage, as in the absence of the Raman nonlinearity ($\mu = 0$), the duration of the soliton-type wave packet decreases down to the duration $\tau_{\rm lim}$ (15). In this case, a decrease in the laser pulse duration is accompanied by a shift of the frequency $\bar\omega$ to $\omega_{\rm lim}$ due to the frequency dependence of the gain. The shift of the carrier frequency becomes particularly strong when the wave packet duration is commensurable with $\tau_{\rm lim}$ (15) and the nonstationarity of the nonlinear response of the medium begins to exert influence. Note that the estimates (19) obtained for these parameters are in good agreement with the results of numerical simulation of the initial Ginzburg–Landau equation with allowance for the Raman nonlinearity (2).

In the opposite limiting case [$\mu \gg \gamma/(4\Omega^3)$], expressions (18) take the form

$$\tau_{\rm lim} \approx \sqrt{\frac{\mu\Omega}{\gamma}}, \quad \bar\omega_{\rm lim} \approx -\Omega + \frac{\gamma}{4\Omega^2\mu}. \tag{20}$$

It can be seen that the minimum duration of a soliton increases with increasing coefficient μ according to the root law. In this case, the centre of the soliton spectrum is concentrated near the boundary of the gain band of the active medium, $\bar\omega_{\rm lim} \approx -\Omega$.

Examples of the phase plane in this limiting case are shown in Figs 1c and 1d. Far from the equilibrium state ($\Omega\tau_{\rm p} \gg 1$), there occurs only a change in the duration of the soliton-like laser pulse at an almost constant carrier frequency.

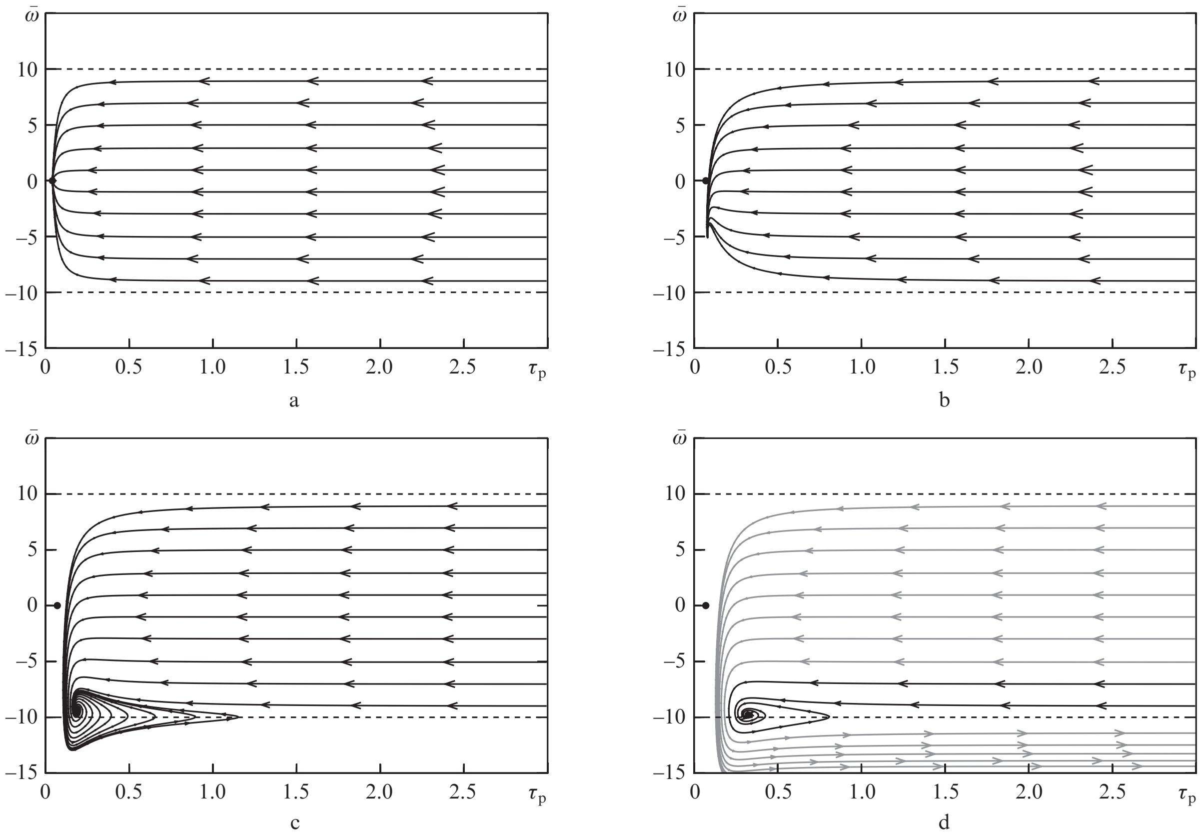

Figure 1. Phase planes $\tau_p, \bar\omega$ for the system of equations (17) constructed at the coefficients (a) $\gamma = 0.3, \mu = 0$; (b) $\gamma = 0.3, \mu = 10^{-4}$; (c) $\gamma = 0.03, \mu = 10^{-4}$; and (d) $\gamma = 0.01, \mu = 10^{-4}$. The horizontal dashed lines indicate the boundaries of the gain band of the active medium ($\Omega = \pm 10$), and the black dots indicate the values of the limiting duration $\tau_{\rm lim}$ (16).

Conversely, at small durations ($\Omega\tau_p \approx 1$), the carrier frequency shifts quite rapidly to the long-wavelength part of the spectrum, i.e. to the edge of the gain band ($\bar\omega \approx -\Omega$) because of the Raman nonlinearity. The duration of the oscillating wave packet at the final stage tends to the value of $\tau_{\rm lim}$ in (20). Moreover, at $\mu\Omega^3/\gamma \gtrsim 3.383$, trajectories appear that envelop the equilibrium state and lead to an increase in the pulse duration (grey curves in Fig. 1d). This case will be considered below in more detail.

Figure 2 shows the results of numerical simulation of the initial Ginzburg–Landau equation (2) with parameters $\gamma = 0.03$, $\Omega = 10$, and $\mu = 10^{-4}$, corresponding to the phase plane in Fig. 1c. At the nonlinear medium input, the distribution of the soliton-type wave packet was set as

$$\Psi(z = 0, \tau) = \frac{\sqrt{2}}{\tau_0} \frac{\exp(i\bar\omega_0\tau)}{\cosh(\tau/\tau_0)} \quad (21)$$

at $\bar\omega_0 = 0$ and $\tau_0 = 10$. Figure 2a shows the dynamics of the self-action of the wave packet envelope in the active fibre. The minimum duration of the wave packet is reached at $z \approx 100$. The inset illustrates the distribution of the wave packet on a larger scale. At the same time, during the shortening of the wave structure, a radiation drop ($z \approx 80$) is observed. Figure 2b shows the dynamics of the wave packet spectrum. It can be seen that the Raman nonlinearity makes the laser pulse spectrum shift to the edge of the gain band ($\bar\omega \approx -\Omega$). The presence of a drop in the time domain explains the irregular character of the wave-field spectrum ($z \approx 80$). However, in the future this irregularity disappears ($z \approx 96$), since the 'dropped' part of the laser pulse escapes rather quickly from the main part of the pulse due to the strong difference in group velocities and is absorbed near the boundaries of the computational domain.

Figures 2c and 2d show the dependence of the maximum amplitude and duration of the wave packet on the evolutionary coordinate z. One can see that a maximum increase in the field amplitude and a maximum decrease in the laser pulse duration are achieved at $z \approx 100$; then, the amplitude decreases and the duration increases. In this case, after a short transient process, the duration and amplitude of the wave packet do not change. Consequently, a soliton solution is excited, in which the duration, amplitude, and carrier frequency do not change. Note that the output pulse duration is approximately 2.6 times longer than the laser pulse duration (16) in the absence of the Raman nonlinearity ($\mu = 0$). Thus, the results of numerical calculations of the initial equation (2) (Fig. 2) are in good agreement with the results of the qualitative analysis of relation (20).

In the case of an even smaller gain ($\gamma \lesssim 0.295\mu\Omega^3$), the dynamics of the system (Fig. 1d) essentially depends on the initial frequency $\bar\omega_0$ of the wave packet (21) injected into the active fibre. In Fig. 1d, two characteristic groups of trajectories in the phase plane are clearly distinguished as functions of the initial frequency $\bar\omega_0$. If the initial frequency is close to the boundary of the gain band (black curves), then the dynamics of the system leads to solutions with a finite pulse duration (trapping into a focus), as in the case of Fig. 1c. Note that with a further decrease in the gain γ, the width of this region will decrease.

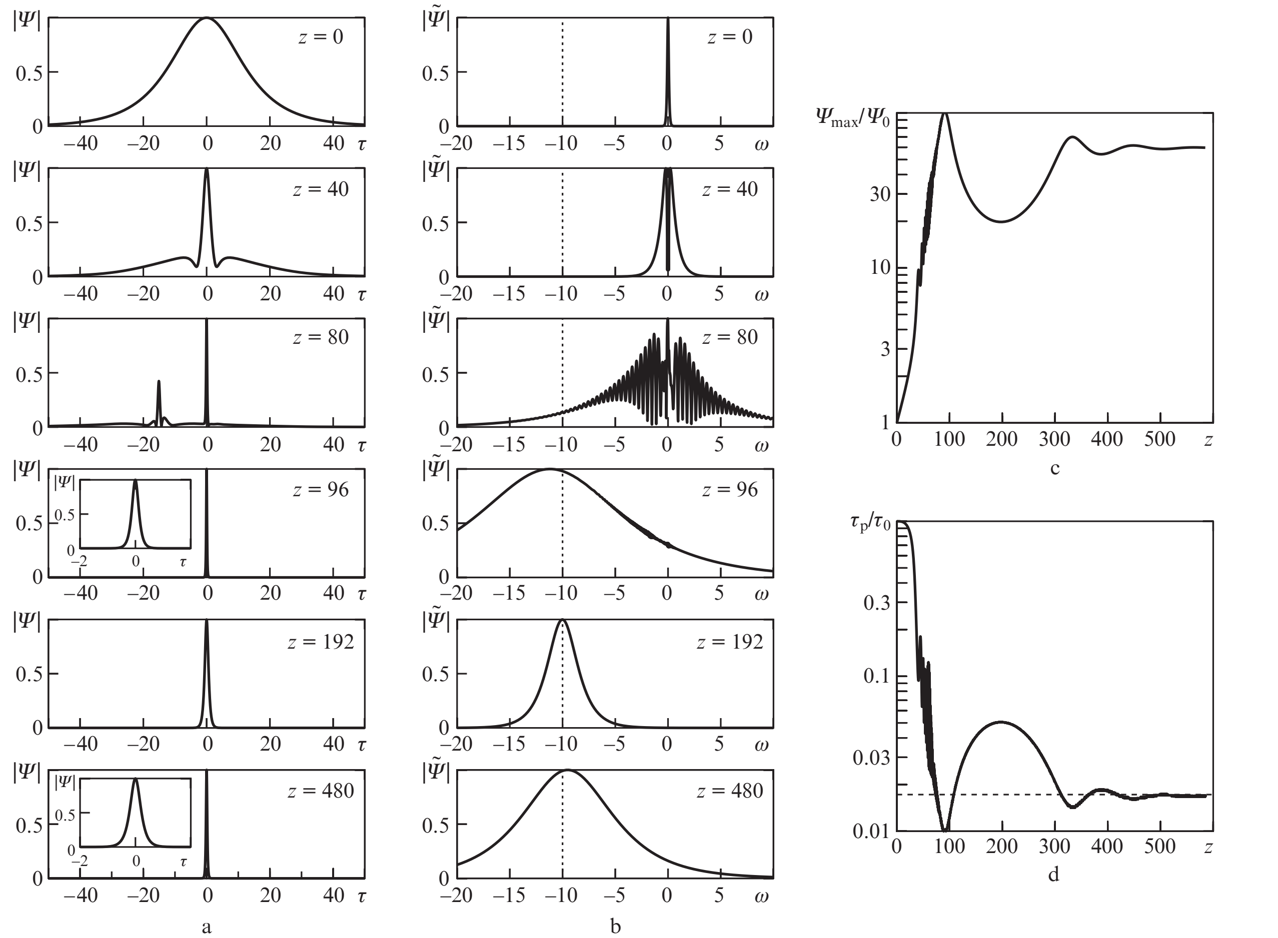

Figure 2. Evolution of (a) the envelope of the wave packet $\Psi(z,\tau)$ and (b) its spectrum $\tilde{\Psi}(z,\omega)$ along the propagation path of length z (the electric field strength and its spectrum are normalised to their maximum values), as well as dependences of (c) the maximum amplitude of the wave packet and (d) the duration of the laser pulse on z. The calculation was performed for $\gamma = 0.03$, $\mu = 10^{-4}$, and $\Omega = 10$. At the input of the nonlinear medium, the initial distribution (21) was set at $\bar{\omega}_0 = 0$ and $\tau_0 = 10$. The horizontal dashed line in Fig. 2d indicates the duration, corresponding to the equilibrium state (18a), and the vertical dashed line in Fig. 2b shows the boundary of the gain band of the active medium.

The results of numerical simulation of the initial Ginzburg–Landau equation (2) are in good agreement with the results of qualitative analysis (Fig. 3). A laser pulse with an initial distribution (21) and parameters $\bar{\omega}_0 = -9$ and $\tau_0 = 10$ was injected to the fibre input. One can see from Fig. 3 that the wave packet frequency remains practically unchanged during an adiabatic decrease in the laser pulse duration (Fig. 3b). In addition, a radiation drop is not observed in the time domain (Fig. 3a). In this case, the output pulse duration (at $z = 2240$) is $4.5 \approx \sqrt{20}$ times larger than the compressed wave packet duration in the absence of the Raman nonlinearity ($\mu = 0$) (16), in full agreement with the estimate in (20). Figures 3c and 3d show the dependences of the maximum amplitude and duration of the wave packet on the propagation path length z. It can be seen that a maximum increase in the field amplitude and a maximum decrease in the laser pulse duration are achieved at $z \approx 900$. In the future, after a short transient process, the considered quantities become stationary.

Let us now analyse the case when the initial frequency of the injected laser pulse is in the centre of the gain band. This corresponds to the phase trajectories in Fig. 1d (grey curves). It follows from this figure that it is impossible to initiate a soliton solution at the final stage. For a complete understanding of the situation, Fig. 4 shows the results on the dynamics of the self-action of the wave packet with the initial time distribution (21) for $\bar{\omega}_0 = 0$ and $\tau_0 = 10$ in the active fibre with parameters $\gamma = 10^{-2}$, $\mu = 10^{-4}$, and $\Omega = 10$. One can see that at the initial stage there is a significant decrease in the laser pulse duration at an almost unchanged carrier frequency (Figs 4a and 4b for $z = 205$), which agrees well with the results of qualitative analysis given above (see Fig. 1d). However, after that, the Raman nonlinearity begins to exert an influence, which leads to a shift in the wave packet spectrum beyond the gain band of the active medium at practically unchanged laser pulse duration. This boundary is shown in Fig. 4b by a dashed line. The subsequent dynamics of the wave packet self-action consists in a substantial increase in the duration and a decrease in the amplitude of the wave packet (Figs 4c and 4d).

Thus, the results of the qualitative analysis carried out within the framework of the variational approach are in good quantitative agreement with the results of numerical simulation performed within the framework of the Ginzburg–Landau equation (2).

4. Self-compression of laser pulses in the framework of the general model

In the previous section, we analysed the self-action of a laser pulse in an active medium when the gain profile in the spectral region $G(\omega)$ was approximated by a parabola [$G(\omega) = \gamma - D\omega^2$].

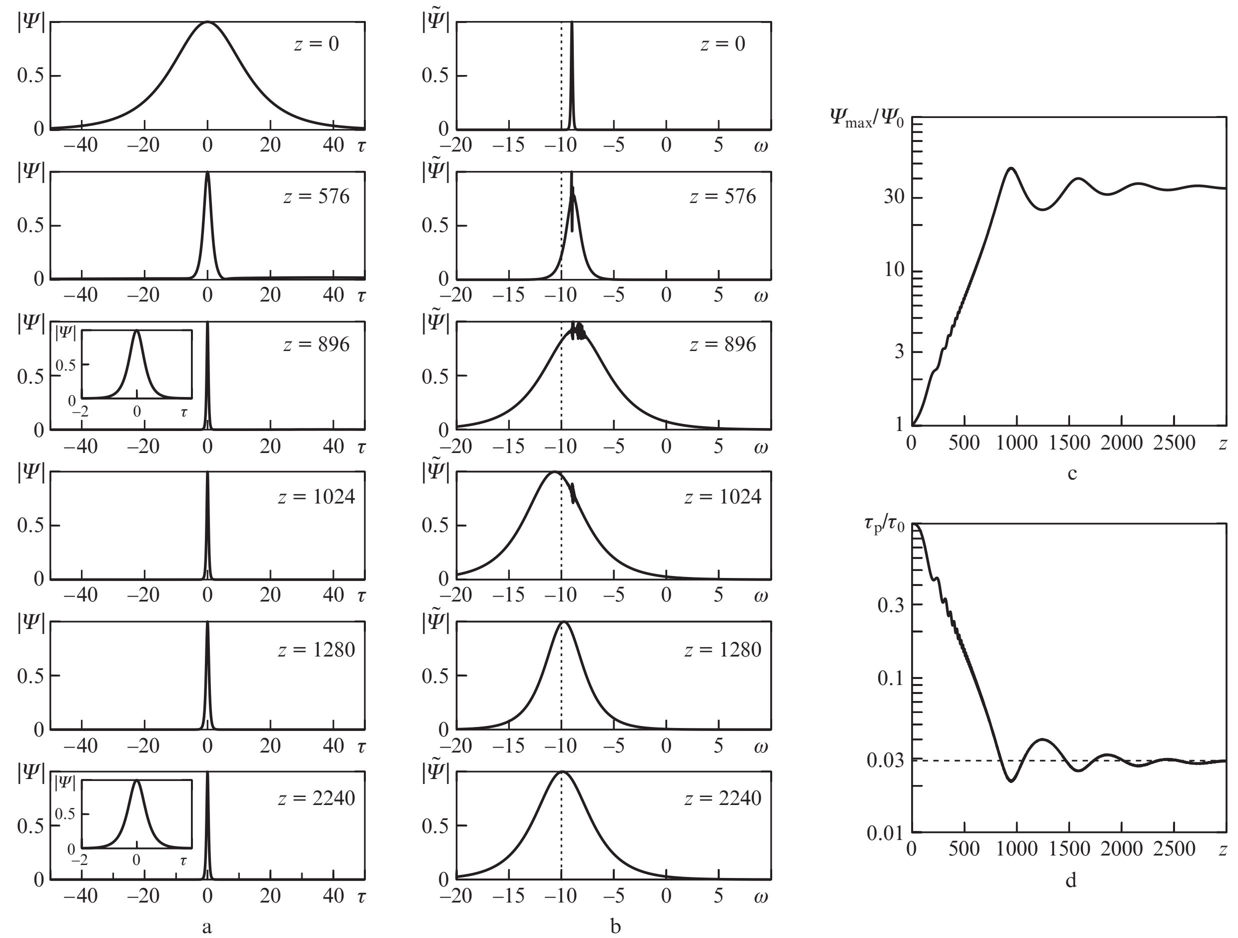

Figure 3. Evolution of (a) the envelope of the wave packet $\Psi(z,\tau)$ and (b) its spectrum $\tilde{\Psi}(z,\omega)$ along the propagation path of length z (the electric field strength and its spectrum are normalised to their maximum values), as well as dependences of (c) the maximum amplitude of the wave packet and (d) the duration of the laser pulse on z. The calculation was performed for $\gamma = 10^{-2}$, $\mu = 10^{-4}$, and $\Omega = 10$. At the input of the nonlinear medium, the initial distribution (21) was set at $\bar{\omega}_0 = -9$ and $\tau_0 = 10$. The horizontal dashed line in Fig. 3d indicates the duration, corresponding to the equilibrium state (18a), and the vertical dashed line in Fig. 3b shows the boundary of the gain band of the active medium.

The suitability of the considered model for describing the gain profile of the medium raises many questions, one of which is the artificial injection of large losses into the spectral wings at $|\omega| > \Omega$. It should be noted that sometimes this model is valid. For example, in practice, losses at the edge of the gain band are additionally introduced into the fibre amplifier to counteract the parasitic signal amplification, while the central part of the laser pulse spectrum is subjected to amplification.

Below, based on an analytical and numerical study of the initial equation (1), we analyse the possibility of adiabatic reduction of the laser pulse duration and determine the characteristic scenarios of the evolution of the wave packet for a sign-constant gain profile. As the gain profile of the active medium $G(\omega)$, we choose a Gaussian function

$$G(\omega) = \gamma \exp\left(-\frac{\omega^2}{\Omega^2}\right) \quad (22)$$

to obtain more illustrative analytical relationships.

For equation (1), the variation of the dissipative function is

$$\delta Q = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} G(\omega)[\tilde{\Psi}(\omega)\delta\tilde{\Psi}^*(\omega) - \tilde{\Psi}^*(\omega)\delta\tilde{\Psi}(\omega)]\mathrm{d}\omega + \mu\int_{-\infty}^{+\infty}\frac{\partial|\Psi|^2}{\partial\tau}\delta|\Psi|^2\mathrm{d}\tau. \quad (23)$$

As a result, the Euler equations (7) for the parameters $a_j = \{W, \tau_p, \beta, \bar{\omega}, q, \theta\}$ of the wave packet (4), describing the self-action of the wave field in an active medium with a Raman nonlinearity, have the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial\bar{\mathcal{L}}}{\partial\dot{a}_j} - \frac{\partial\bar{\mathcal{L}}}{\partial a_j} = \mu\int_{-\infty}^{+\infty}\frac{\partial|\Psi|^2}{\partial\tau}\frac{\partial|\Psi|^2}{\partial a_j}\mathrm{d}\tau + \frac{\mathrm{i}\gamma}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\frac{\omega^2}{\Omega^2}\right)\left[\tilde{\Psi}(\omega)\frac{\partial\tilde{\Psi}^*(\omega)}{\partial a_j} - \tilde{\Psi}^*(\omega)\frac{\partial\tilde{\Psi}(\omega)}{\partial a_j}\right]\mathrm{d}\omega. \quad (24)$$

Calculating the integrals on the right-hand side of (24), we arrive at the following system of ordinary differential equations for the parameters of the localised structure (4):

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{2W\tau_p\Omega\gamma}{\sigma}\exp(-\tau_p^2\bar{\omega}^2/\sigma^2), \quad (25a)$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}z} = \frac{1}{\tau_p^4} - 4\beta^2 - \frac{W}{\sqrt{8\pi}\,\tau_p^3} - \frac{4\gamma\tau_p\beta\Omega}{\sigma^5} \times(\sigma^2 - 2\tau_p^2\bar{\omega}^2)\exp(-\tau_p^2\bar{\omega}^2/\sigma^2), \quad (25b)$$

$$\frac{\mathrm{d}\tau_p}{\mathrm{d}z} = 4\beta\tau_p - \frac{\tau_p^2\Omega\gamma}{\sigma^5}(4\tau_p^6\beta^2\Omega^2 - \tau_p^2\Omega^2 - 8\tau_p^6\beta^2\bar{\omega}^2 + 2\tau_p^2\bar{\omega}^2 + 16\tau_p^8\beta^4 - 1)\exp(-\tau_p^2\bar{\omega}^2/\sigma^2), \quad (25c)$$

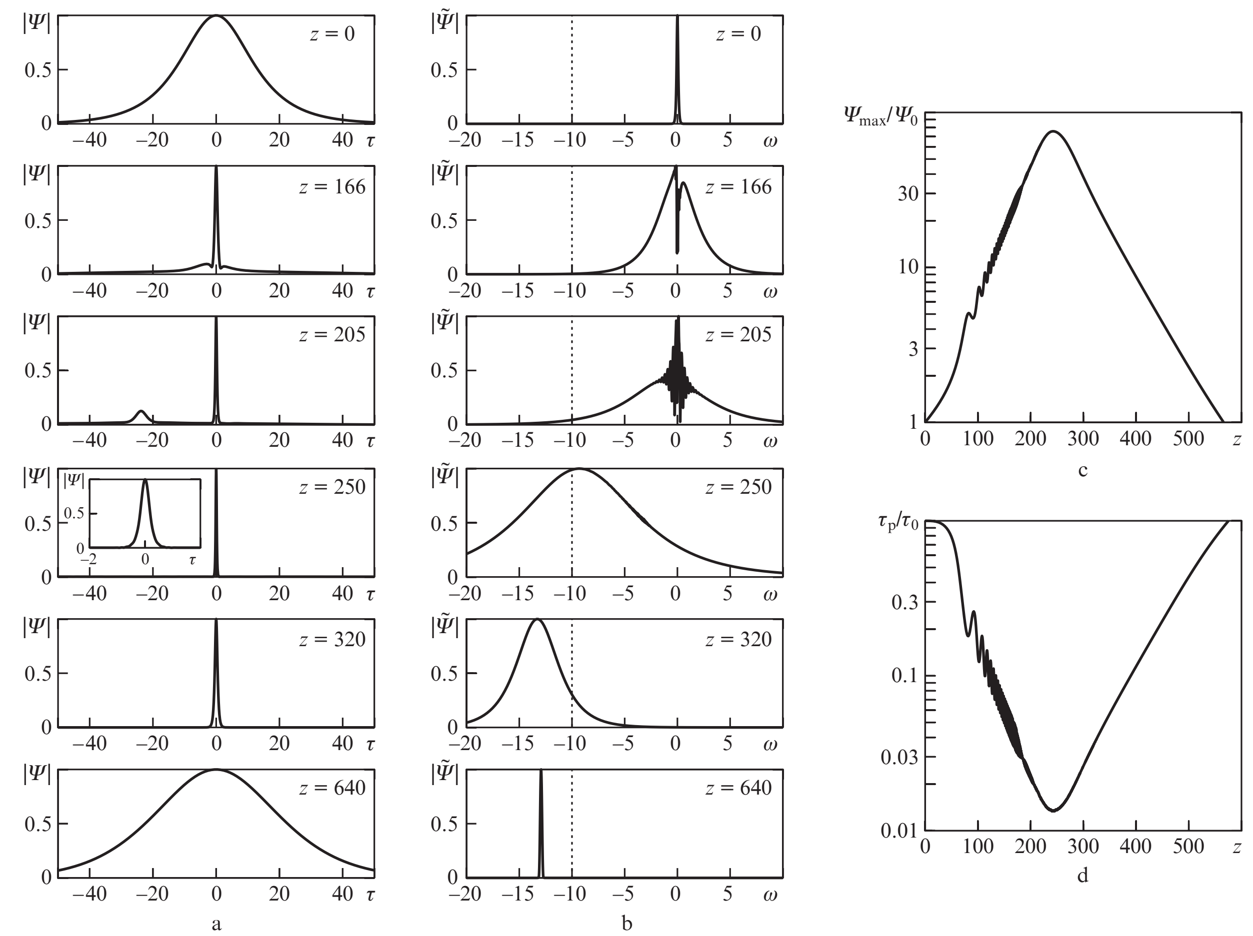

Figure 4. Evolution of (a) the envelope of the wave packet $\Psi(z,\tau)$ and (b) its spectrum $\tilde{\Psi}(z,\omega)$ along the propagation path of length z (the electric field strength and its spectrum are normalised to their maximum values), as well as dependences of (c) the maximum amplitude of the wave packet and (d) the duration of the laser pulse on z. The calculation was performed for $\gamma = 10^{-2}$, $\mu = 10^{-4}$, and $\Omega = 10$. At the input of the nonlinear medium, the initial distribution (21) was set at $\bar{\omega}_0 = 0$ and $\tau_0 = 10$. The vertical dashed line in Fig. 3b shows the boundary of the gain band of the active medium.

$$\frac{d\bar{\omega}}{dz} = -\frac{\mu W}{\sqrt{2\pi}\,\tau_p^3} - \frac{2\tau_p\bar{\omega}\Omega\gamma}{\sigma^3}(1 + 4\tau_p^4\beta^2)\exp(-\tau_p^2\bar{\omega}^2/\sigma^2), \quad (25d)$$

$$\frac{dq}{dz} = \frac{4\tau_p^5\beta\bar{\omega}\Omega\gamma}{\sigma^3}\exp(-\tau_p^2\bar{\omega}^2/\sigma^2) - 2\bar{\omega}, \quad (25e)$$

where $\sigma = (\tau_p^2\Omega^2 + 4\tau_p^4\beta^2 + 1)^{1/2}$. As in the case of a qualitative analysis of the Ginzburg–Landau equation, equation (25e) for the velocity of the maximum of the intensity of the wave packet is isolated from the rest of the system of equations (25).

A further simplification of the qualitative analysis of the self-action of a laser pulse in an active medium with a Gaussian gain profile of form (22) can consist in considering soliton-like wave structures. In this case, as in Section 3, from equations (25b) and (25c), we can find the relationship between the laser pulse energy and the duration, analogous to (11):

$$W \approx \frac{\sqrt{8\pi}}{\tau_p}. \quad (26)$$

In this case, the expression for frequency modulation β is analogous to (12):

$$\beta \approx -\frac{\gamma}{4\tau_p^2\Omega^2}. \quad (27)$$

As before, we assume that the frequency modulation is small at the scale of the wave packet ($\beta\tau_p^2 \ll 1$). As a result, we arrive at the following system of equations determining a decrease in the duration τ_p and the shift of the central frequency $\bar{\omega}$ of the soliton-like wave packet:

$$\frac{d\tau_p}{dz} = -2\frac{\tau_p^2\Omega\gamma}{\sqrt{1 + \tau_p^2\Omega^2}}\exp\left[-\frac{\tau_p^2\bar{\omega}^2}{(1 + \tau_p^2\Omega^2)^2}\right], \quad (28a)$$

$$\frac{d\bar{\omega}}{dz} = -\frac{2\mu}{\tau_p^4} - \frac{2\bar{\omega}\Omega\gamma\tau_p}{(1 + \tau_p^2\Omega^2)^{3/2}}\exp\left[-\frac{\tau_p^2\bar{\omega}^2}{(1 + \tau_p^2\Omega^2)^2}\right]. \quad (28b)$$

The phase planes for this system of equations are shown in Fig. 5.

Figure 5a shows a change in the duration and carrier frequency of the laser pulse in the absence of nonstationarity of the nonlinear response of the medium ($\mu = 0$). It can be seen that unlike the above-considered case analysed in the framework of the Ginzburg–Landau equation, the minimum duration of the laser pulse is not limited by the gain band (see Fig. 1a). This adiabatic unlimited reduction in the wave packet duration can be understood from the following simple considerations. The central part of the pulse spectrum that falls into the gain band of the active medium exponentially increases, and then the accumulated energy is redistributed

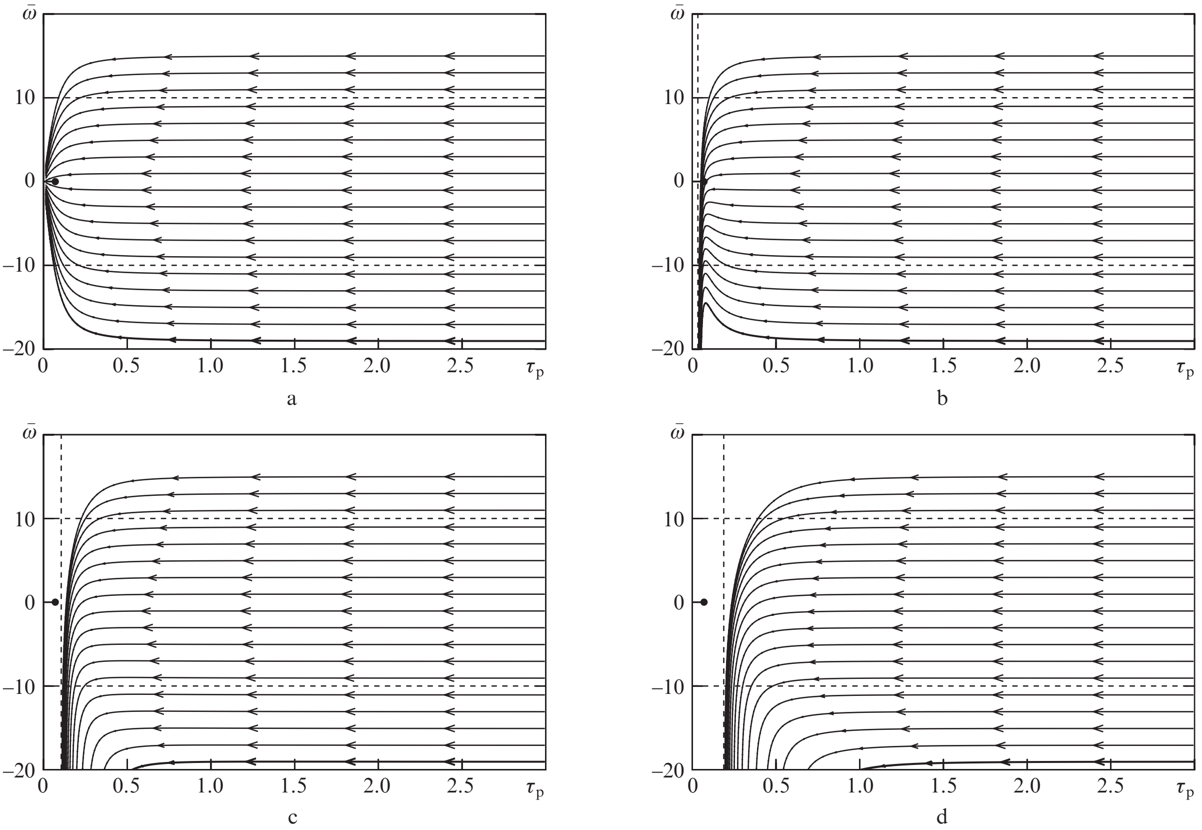

Figure 5. Phase planes $\tau_p, \bar{\omega}$ for the system of equations (17) constructed at the coefficients (a) $\gamma = 0.01$, $\mu = 0$, (b) $\gamma = 0.01$, $\mu = 10^{-6}$, (c) $\gamma = 0.01$, $\mu = 10^{-4}$, and (d) $\gamma = 0.001$, $\mu = 10^{-4}$. The vertical dashed lines indicate the estimates of the minimum laser pulse duration obtained within the framework of the initial equation (1), the horizontal dashed lines show the boundaries of the gain band of the active medium, and the black dots displaty the estimates of the minimum duration of the wave packet $\tau_{\rm lim}$ (16) obtained within the framework of the Ginzburg–Landau equation at $\mu \ll \gamma/(4\Omega^3)$.

throughout the entire spectrum of the laser pulse due to the action of the linear dispersion of the medium in such a way that the laser pulse remains a soliton. It should be noted that the possibility of obtaining laser pulses with a duration less than the inverse gain bandwidth has been discussed in various papers (see, for example, [32]). In this case, if the initial frequency $\bar{\omega}_0$ of the laser pulse injected into the fibre is different from the carrier frequency of the gain band ($\bar{\omega} \neq 0$), one can observe a frequency shift $\bar{\omega}$ tending to zero due to the inhomogeneity of the gain band (22) (Fig. 5a). This shift is described by the second term on the right-hand side of equation (28b).

To determine the law of a decrease in the wave packet duration τ_p, depending on the evolutionary variable z, we consider the case $\bar{\omega}_0 = 0$. Here the position of the carrier frequency of the laser pulse will not change, since $\bar{\omega}(z) = 0$ is a solution of equation (28b) at $\mu = 0$. As a result, we can write down the solution for the laser pulse duration:

$$2\sqrt{1 + \frac{1}{\tau_p^2 \Omega^2}} - 2\sqrt{1 + \frac{1}{\tau_0^2 \Omega^2}} + \ln\frac{\sqrt{1 + 1/(\tau_p^2 \Omega^2)} - 1}{\sqrt{1 + 1/(\tau_p^2 \Omega^2)} + 1}$$

$$- \ln\frac{\sqrt{1 + 1/(\tau_0^2 \Omega^2)} - 1}{\sqrt{1 + 1/(\tau_0^2 \Omega^2)} + 1} = 4\gamma z, \qquad (29)$$

where τ_0 is the initial duration of the wave packet.

In the case of long pulses ($\tau_0 \Omega \gg \tau_p \Omega \gg 1$), taking into account the small parameter $1/(\Omega \tau_p)$, we obtain

$$\tau_p \approx \tau_0 \exp(-2\gamma z). \qquad (30)$$

This means that the laser pulse duration decreases exponentially at the initial stage. In the other limiting case ($\tau_p \Omega \ll 1$), we have

$$\tau_p \propto \frac{1}{2\Omega \gamma z}. \qquad (31)$$

Thus, at the final stage, the laser pulse duration will decrease to zero in accordance with the power law, i.e., rather slowly compared with the decrease in accordance with the exponential law.

Let us now consider the case $\mu \neq 0$. It follows from Fig. 5b that even at a small value of the coefficient μ there is a qualitative difference from the case $\mu = 0$. Taking into account the nonstationarity of the nonlinear response leads to the limitation of the minimum laser pulse duration $\tau_{\rm lim}$. One can see that at the initial stage, when the Raman nonlinearity of the medium does not affect the dynamics of the wave field, an adiabatic decrease in the wave packet duration occurs at a virtually unchanged carrier frequency. Then, the carrier frequency of the pulse $\bar{\omega}$ will shift to zero because of the frequency dependence of the gain $G(\omega)$. At the final stage, the Raman response stops decreasing the laser pulse duration due to a

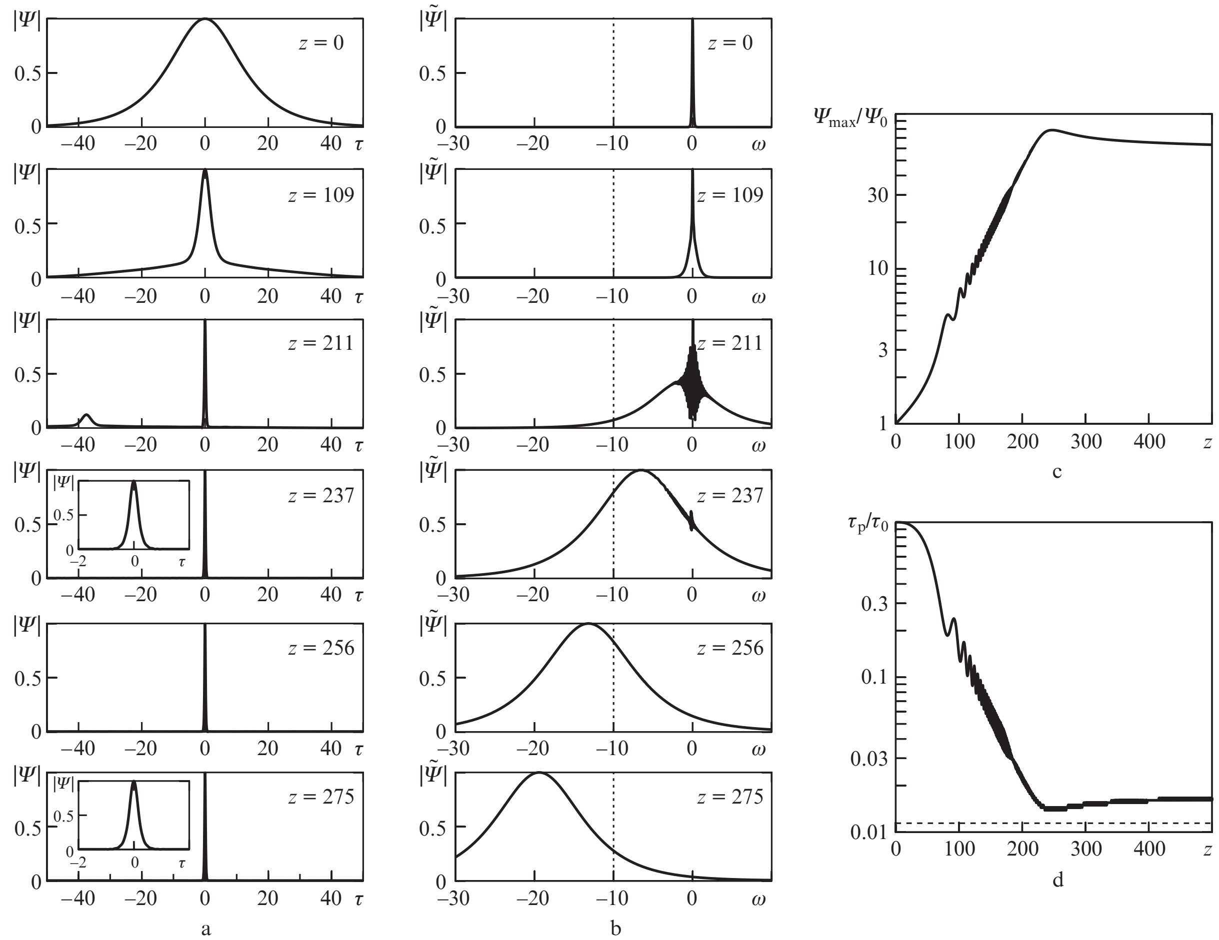

Figure 6. Evolution of (a) the envelope of the wave packet $\Psi(z,\tau)$ and (b) its spectrum $\tilde{\Psi}(z,\omega)$ along the propagation path of length z (the electric field strength and its spectrum are normalised to their maximum values), as well as dependences of (c) the maximum amplitude of the wave packet and (d) the duration of the laser pulse on z. The calculation was performed for $\gamma = 10^{-2}$, $\mu = 10^{-4}$, and $\Omega = 10$. At the input of the nonlinear medium, the initial distribution (21) was set at $\bar{\omega}_0 = 0$ and $\tau_0 = 10$. The horizontal dashed line in Fig. 6d indicates the duration, corresponding to the equilibrium state (36), and the vertical dashed line in Fig. 3b shows the boundary of the gain band of the active medium.

significant shift in the wave field spectrum to the long-wavelength region. It can be seen from Figs 5b–5d that the minimum value of the compressed laser pulse duration increases with decreasing gain γ at a constant coefficient.

We obtain an estimate of the minimum duration $\tau_{\rm lim}$ of the wave packet as a function of the parameters of the problem. To do this, we turn to the truncated system of equations (28). Let a laser pulse with initial parameters $\tau_0\Omega \gg 1$ and $\bar{\omega}_0 = 0$ be injected to the input of a nonlinear medium. Then equation (28a) reduces to equation

$$\frac{d\tau_p}{dz} \approx -2\gamma\tau_p \tag{32}$$

and has a solution

$$\tau_p(z) = \tau_0 \exp(-2\gamma z). \tag{33}$$

Simultaneously, the carrier frequency is rapidly shifted down the spectrum due to the nonstationarity of the nonlinear response of the medium in accordance with equation (28b). In the case under consideration, the latter reduces to the equation

$$\frac{d\bar{\omega}}{dz} = -\frac{2\mu}{\tau_p^4}. \tag{34}$$

Substituting the expression for the wave packet duration (33) into equation (34), we find the solution for the carrier frequency:

$$\bar{\omega}(z) = -\frac{\mu}{4\gamma\tau_0^4}\exp(8\gamma z) \equiv -\frac{\mu}{4\gamma\tau_p^4}. \tag{35}$$

The amplification stops and the approximate equation (32) becomes inapplicable when the carrier frequency is shifted beyond the gain band, that is, at $\bar{\omega} \approx \Omega$. This gives an estimate of the limiting duration of the compressed laser pulse as a function of the medium parameters:

$$\tau_{\rm lim} = \left(\frac{\mu}{4\gamma\Omega}\right)^{1/4}. \tag{36}$$

In Fig. 5 vertical dashed lines indicate the minimum duration of the compressed pulse. It can be seen that the values obtained are in good agreement with the results of a numerical analysis of the phase plane for the system of equations (28). Note that if the carrier frequency of the wave packet injected into the fibre is further shifted up the spectrum, it is possible to slightly increase the compression ratio of the laser pulse.

Next, let us turn to the results of numerical simulation of the initial equation (1) with the Gaussian gain profile (22).

Figure 6 shows the dynamics of the self-action of the wave packet in an active fibre with parameters $\gamma = 10^{-2}$, $\mu = 10^{-4}$, and $\Omega = 10$. A laser pulse with an initial distribution (21) with parameters $\bar{\omega}_0 = 0$ and $\tau_0 = 10$ is injected to the fibre input. Figure 6a shows the dynamics of the wave packet, and Fig. 6b demonstrates the dynamics of the wave-field spectrum. One can see that at the initial stage ($z \approx 200$), there is an adiabatic decrease in the laser pulse duration in the time domain. Simultaneously, in the frequency domain, the spectrum is uniformly broadened in both directions at a constant carrier frequency of the laser pulse. At the second stage ($z > 200$), the further shortening of the laser pulse ceases and there is only a shift of the spectrum centre down the spectrum. Figure 6c and 6d show the dependence of the maximum amplitude and duration of the wave packet on the propagation path length z. One can see that the maximum increase in the field amplitude and the maximum decrease in the laser pulse duration are achieved at $z \approx 200$. Then, the quantities in question reach stationary values. It should be noted that the obtained estimate of the minimum wave packet duration (36) is in good agreement with the results of numerical simulations. In Fig. 6d, the horizontal dashed line indicates this estimate.

Thus, the results of the qualitative analysis based on the variational approach are in good quantitative agreement with the results of the numerical simulation performed within the framework of the initial equation (1).

5. Conclusions

The influence of the nonlinear response nonstationarity on the self-compression of soliton-like laser pulses during their propagation in active fibres with a finite gain bandwidth is analytically and numerically investigated. The variational approximation is generalised to the case of the description of the nonlinear propagation of wave packets in nonconservative systems with an arbitrary dependence of the gain profile on the frequency. The variational approach is used to analyse qualitatively the self-action of the wave packet in the system. The variational approximation makes it possible to reduce the partial differential equation to a closed system of ordinary differential equations for the characteristic parameters of a solitary laser pulse having a Gaussian distribution. Their analysis allows one to classify the main regimes of the evolution of the wave field and to determine the minimum achievable duration of the laser pulse during self-compression as a function of the parameters of the problem.

In Section 3, the self-action of laser pulses profile is analytically and numerically studied in the framework of the Ginzburg–Landau equation corresponding to a parabolic approximation of the gain profile. Apart from gain, the Ginzburg–Landau equation describes also diffusion of the laser pulse in the fibre, i.e., the attenuation of the wave field outside the gain band. An analysis of the system of equations for the parameters of a solitary pulse show the existence of stable soliton-type structures that are 'pressed' in the spectral region to the boundary of the gain band of the active medium. The minimum duration of the soliton and its frequency shift due to nonstationarity of the nonlinear response are determined as a function of the parameters of the problem. Numerical modelling shows good quantitative agreement with analytical estimates.

An analysis of the self-action of laser pulses in the framework of the NLSE with a constant-sign gain profile is considered in Section 4. However, in this case the system of equations for the parameters of a solitary pulse does not have a stationary solution. An exception is the case of absence of the Raman nonlinearity, when it is possible to reduce the laser pulse duration to zero in accordance with the power law. The presence of an arbitrarily small nonstationary nonlinear response changes the situation qualitatively. This response leads to a significant shift of the carrier frequency of the pulse beyond the gain region, which stops the compression of the laser pulses. An expression for the minimum duration of the soliton-like laser pulse is obtained as a function of the parameters of the problem, which agrees well with the results of numerical simulation of the initial equation.

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