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# Chaotic vortex filaments in a Bose–Einstein condensate and in superfluid helium

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***Abstract.*** **A statement of the quantum turbulence problem in both a Bose–Einstein condensate (BEC) and superfluid helium is formulated. In superfluid helium use is made of a so-called vortex filament method, in which quantum vortices are represented by string-like objects, i.e. vortex lines. The dynamics of the vortex lines is determined by deterministic equations of motion, supplemented by random reconnections. Unlike He II, the laws of the dynamics of quantum vortices in BEC are based on the nonlinear Schrödinger equation. This makes it possible to obtain a microscopic description of the collision of vortices, the structure of a vortex filament, etc. A comparative analysis of these complementary approaches is carried out. It is shown that there are some features that do not automatically transfer the results obtained for BEC to vortices in He II and vice versa.**

***Keywords:*** *Bose–Einstein condensate, quantum vortices, superfluid turbulence, topological defects.*

## 1. Introduction

The investigation of quantum vortices initiated by the discovery of superfluidity in liquid helium has received a new powerful impetus after the discovery of a Bose–Einstein condensate (BEC) for ultracold atomic gases (see, for example, reviews [1, 2]) and observations of a new type of quantum liquids with experimentally controlled parameters. A critically important circumstance is that, unlike superfluid helium, for BEC there is a microscopic theory that describes the dynamics of the system and, in particular, the structure and evolution of quantum vortices. Theoretical aspects of the BEC dynamics were set forth in a well-known book by Pitaevskii and Stringari [3]. In the domestic literature we should mention papers by Chapovsky [4], Likhanova et al. [5], and Taichenachev et al. [6].

As in superfluid helium, of special, possibly leading, place is the study of vortex states in BEC, in particular, the examination of the dynamics of a chaotic tangle of vortices, or the so-called quantum turbulence (QT). The presence of microscopic theory and expanded experimental possibilities make BEC a more advanced system that allows a deeper and more thorough research of QT problems. However, the question arises as to how far the results on chaotic vortices obtained for BEC are applicable to superfluid helium. In this paper, a statement of the QD problem in both BEC and superfluid helium (Sections 2 and 3) is formulated and a comparative analysis of the various approaches is carried out (Section 4). The Conclusion discusses the degree of compatibility of both approaches.

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## 2. Quantum turbulence in superfluid helium

Based on quantum mechanical properties of a superfluid liquid, Feynman [7] and Onsager [8] predicted that the He II vortex motion is realised in a very specific way. In particular, they suggested that one-dimensional singularities, or vortex threads, arise in helium, on which the condition $\nabla\times\boldsymbol{v}_{\rm s} = 0$ is violated. A circular motion or circulation of a superfluid helium component is possible around these singularities. The quantum mechanical properties of a superfluid liquid impose restrictions on the circular motion. For example, the circulation of the superfluid component velocity takes only certain, quantised values:

$$\int \boldsymbol{v}_{\rm s} {\rm d}\boldsymbol{l} = j\kappa, \tag{1}$$

where the integral is calculated along any contour enclosing the filament, and the vorticity quantum is

$$\kappa = 2\pi\hbar/m_{\rm He} = 9.97\times10^{-4}\,{\rm cm^2\,s^{-1}}. \tag{2}$$

The vorticity field $\boldsymbol{\omega}(\boldsymbol{r}) = \nabla\times\boldsymbol{v}_{\rm s}$ is such that $\boldsymbol{\omega}(\boldsymbol{r}) = 0$ and $\boldsymbol{\omega}(\boldsymbol{r}) = \infty$ outside and on the line, respectively. Formally, the vorticity field $\boldsymbol{\omega}(\boldsymbol{r})$ can be represented as follows:

$$\boldsymbol{\omega}(\boldsymbol{r}) = \nabla\times\boldsymbol{v}_{\rm s} = \kappa\int {\rm d}\boldsymbol{s}\delta[\boldsymbol{r} - \boldsymbol{s}(\xi, t)], \tag{3}$$

where the integration takes place along the filament.

Thus, quantum vortices behave absolutely identical to thin vortex tubes studied in classical hydrodynamics, except that the latter are considered only as a convenient and fruitful mathematical model. A vortex filament can be described in parametric form by the function $\boldsymbol{s}(\xi, t)$, where $\boldsymbol{s}$ is the radius vector of the point of the line, and the parameter $\xi$ 'recalculates' the points of the line; often, the value of $\xi$ is the arc length (Fig. 1). The set of lines $\{\boldsymbol{s}(\xi, t)\}$ evolves, obeying the equations of motion and boundary conditions. If the vorticity field $\boldsymbol{\omega}(\boldsymbol{r})$ is known, we can easily write the expression for the velocity of the elements of the vortex filament $\dot{\boldsymbol{s}}_i(\xi, t)$ induced by the vortex line. This expression using the Biot–Savart law will have the form:

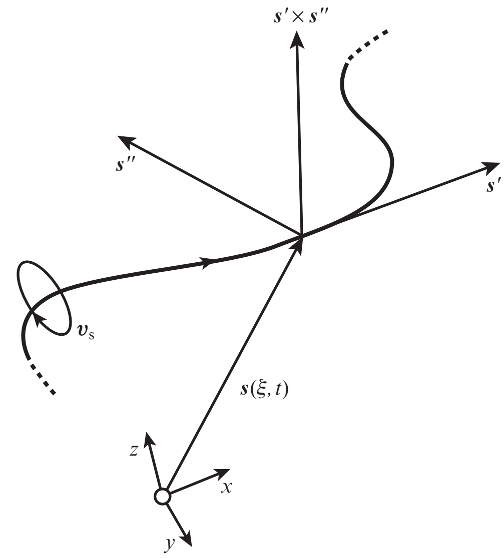


**Figure 1.** Schematic representation of a vortex line in an arbitrary coordinate system. Each point of the vortex line $s(\xi, t)$ is determined by the Cartesian coordinates $x$, $y$, $z$ and the parameter $\xi$ along the line. The vectors $s'$, $s''$ and $s' \times s''$ are the tangential vector, the local curvature vector, and the binormal vector, which coincides with the direction of the locally induced velocity $\dot{s}$ of the point of the vortex line $s(\xi, t)$.

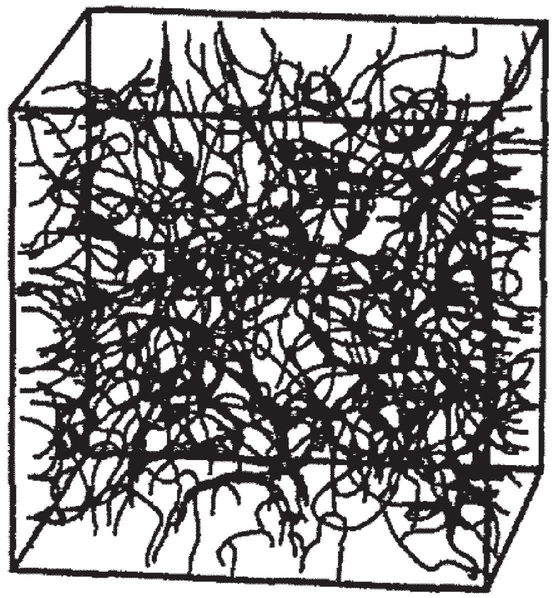

**Figure 2.** Vortex tangle in a He II counter flow (see the text). The figure is borrowed from [10] (Fig. 13).

$$\dot{s}_i(\xi, t) = \frac{\kappa}{4\pi} \int \frac{[s(\xi', t) - s(\xi, t)] \times s'_{\xi'}}{|s(\xi', t) - s(\xi, t)|^3} \, d\xi' . \tag{4}$$

In addition to deterministic evolution, there is a stochastic element of dynamics, i.e. random collisions of filaments with each other, followed by reconnection. The study of vortices and vortex dynamics within the framework of the described procedure is called the vortex filament method.

The term 'quantum turbulence' was introduced by Feynman in his fundamental work [7], where he explained the results obtained by Gorter and Mellink [9]. Gorter and Mellink observed a sharp increase in the temperature difference in a He II counter flow, when the velocity exceeded a certain, rather small, value. Feynman associated the crisis of Gorter and Mellink with the appearance of an unordered set of quantum vortex lines, or a vortex tangle in the system, which impedes the entropy-transferring flow of the normal component. Feynman also proposed a qualitative scenario for the development of the vortex tangle. In particular, he described the mechanisms that lead to an increase in the density of the vortex filaments, and proposed the laws of a decay of the vortex tangle. Feynman's ideas were developed in a series of experimental works by Vinen, who constructed a phenomenological theory of superfluid turbulence. The first clear confirmation of Feynman's ideas was obtained three decades later in the work of Schwarz [10], who demonstrated the appearance of a vortex tangle in direct numerical simulation. An example of a vortex coil developing in a He II counter flow from initially smooth (six) vortex rings is shown in Fig. 2. After almost another three decades, a series of experiments was performed on the visualisation of helium flows, in which the authors observed many disordered vortices (see, for example, review [11]).

At present, the QT theory is an actively developing field, which has many applications both in the theory of superfluidity and in other spheres of physics. Examples include classical turbulence theories [11–13], cosmic strings [14], dislocations in solids [15] and phase transitions [16]. The QT concept is also used in the study of quark–gluon plasma [17] and in the physics of neutron stars [18].

Another justification for interest in QT, attractive to theorists, is that the theory of superfluid turbulence is an elegant task of statistical physics for the set of string-like objects with nonlinear and nonlocal interactions. An additional complexity is the reconnection, leading to the merging or splitting of the vortex loops (see review [19]).

## 3. Quantum turbulence in a Bose–Einstein condensate

Because of the absence of a rigorous microscopic theory for He II, the results related to vortices in superfluid helium have a phenomenological character. This concerns the equation of motion, the structure of the nucleus, the reconnection procedure, etc. Therefore, there are various artificial recipes, such as 'reconnection ansatz' [19]. In contrast, the dynamics laws for vortices in BECs are based on the Gross–Pitaevskii equation (GPE) for the macroscopic wave function $\psi(\mathbf{r}, t)$, which is nothing but a variant of the nonlinear Schrödinger equation. The GPE was used by different authors to study turbulence (see review [20]). In this regard, many of the listed problems have solutions that are widely used to interpret similar phenomena in superfluid helium.

Following [21], we write the GPE for the macroscopic wave function $\psi(\mathbf{r}, t)$ in the form

$$\frac{\hbar}{m} \frac{\partial \psi}{\partial t} = -(\Lambda + \mathrm{i}) \frac{\delta H(\psi)}{\delta \psi^*} . \tag{5}$$

The Ginzburg–Landau free energy functional $H\{\psi\}$ can be represented in the form

$$H\{\psi\} = \int d^3\mathbf{r} \Big[ \frac{\hbar^2}{2m^2} |\nabla \psi|^2 - \frac{\mu}{m} |\psi|^2 +$$

$$+\frac{U_0}{2m}|\psi|^4+V_{\rm ext}(\boldsymbol r)\,|\,\psi(\boldsymbol r)\,|^2\Bigg]. \tag{6}$$

A term with a coefficient $\Lambda$ describes relaxation processes at finite temperatures, and, consequently, evolution must be of dissipative character. An external potential $V_{\rm ext}(\boldsymbol r)$ (for example, a potential of confinement of BEC atoms in a trap) is introduced into the expression for the energy of the system. The quantity $U_0=\int U(\boldsymbol r){\rm d}\boldsymbol r$ is the interaction amplitude of two particles [$U(\boldsymbol r)$ is the real two-particle interaction potential of atoms]. In this case, the chemical potential is $\mu = nU_0$ ($n$ is the concentration of particles in the condensate).

The vortex lines (more precisely, the central lines of the vortex) are the geometric locus of the intersection points of the surfaces ${\rm Re}\,\psi(\boldsymbol r,t)=0$ and ${\rm Im}\,\psi(\boldsymbol r,t)=0$ (Fig. 3). Thus, following the evolution of the zeros of the function $\psi(\boldsymbol r,t)$, we can describe the evolution of vortices. Due to the fact that the vortex lines are a set of points involving the vanishing of the macroscopic wave function, they are also called topological defects.

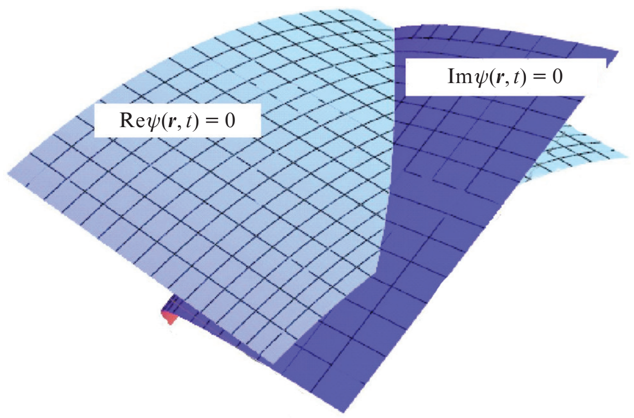


**Figure 3.** (Colour online) Schematic representation of the vortex line as a topological defect. The vortex line is the geometric locus of points where the values of the macroscopic wave function (its real and imaginary parts) vanish.

The Gross–Pitaevskii equation corresponds to the case of weak external perturbations and asymptotically weak atomic interactions. Therefore, in describing turbulence (which corresponds to strong perturbations), this equation can provide only qualitative results. The general picture, which follows from the numerical simulation of the GPE, is as follows. With strong perturbations, quantum vortices appear that evolve, collide, join with each other (or split), and finally form an entangled vortex tangle. The decay of turbulence occurs due to the appearance of Kelvin waves and phonon emission. The importance of vortex reconnection was emphasised by Svistunov [22]. Numerical simulation based on the GPE was considered by Kobayashi and Tsubota [23].

The method for producing highly nonequilibrium states of atoms was developed by Yukalov (see, for example, [24, 25]). The idea of this method is to modulate the trap potential through a nonstationary, inhomogeneous and anisotropic perturbation. When the trap rotates at a normal speed, separate vortices must appear. At a higher rotation speed, there are many vortices that form the Abrikosov lattice. In contrast, if the nonstationary field modulating the trap potential, like Yukalov's method [24–26], does not have a fixed rotation axis, then vortices and anti-vortices randomly located in space are produced, so that the vortex system becomes a mixture of such random vortices [27]. The main idea is that the perturbing alternating potential does not have a single fixed rotation axis. For example, a modulated field can have two dielectric axes. It is also possible, instead of modulating the trap potential, to modulate the atomic scattering length by using the Feshbach resonance methods with an oscillating magnetic field [25, 28].

In the case of applying an external variable potential to a trap of Bose-condensed atoms, many coherent topological modes are generated that decay into vortices with quantised vorticity, since they are the most energetically stable modes [29]. When the number of generated vortices becomes sufficiently large, they form a random tangle typical of QT. The generation of QT of trapped Bose gases by the nonstationary modulation of the trap potential was experimentally realised by the Bagnato group [30]. The method for visualising vortex filaments in weakly interacting Bose–Einstein gases consists in measuring the optical density, the lines with an abnormally low density being associated with vortex filaments (Fig. 4). A description of all the dynamic regimes, starting from an equilibrium Bose gas to producing individual vortices, and then generating QT, is given in [30, 31].

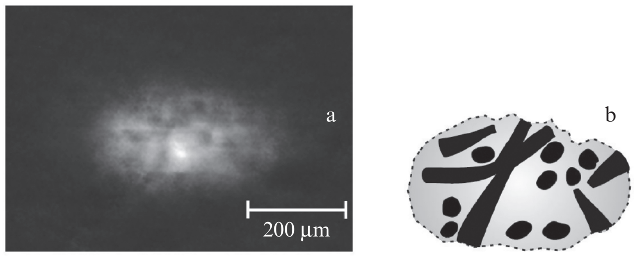


**Figure 4.** (a) Atomic optical density in BEC after 15 ms of free expansion, showing the presence of vortex structures (dark regions; vortex filaments spread all around the sample and resemble the vortex tangle regime), and (b) schematic diagram showing the inferred distribution of vortices as obtained from the image in Fig. 4a. The figure is borrowed from [30].

## 4. Comparative analysis of the two approaches

One of the key questions is whether the two approaches are identical and whether the results obtained for BEC can be used to explain the results in the case of superfluid helium and vice versa. As discussed above, because of the absence of a rigorous microscopic theory for He II, the conclusions concerning the dynamics of vortices in superfluid helium have a phenomenological character. In contrast, the dynamics laws for vortices in BEC are based on the widely applied Gross–Pitaevskii theory. Therefore, many problems of vortex dynamics in BEC have solutions that can be used (with some reservations) to interpret similar phenomena in superfluid helium [32].

The representation of vortex filaments in helium in the form of infinitely thin lines gives an indisputable advantage associated with computer calculations. Indeed, in the case of three-dimensional computations with a chosen step of the spatial partition, or the number $N$ of nodes connected to it, computer costs grow as $N^3$ for BEC. At the same time, in studying the dynamics of vortex lines in superfluid helium, computer costs grow much slower, namely, as $N \ln N$. This allows one to calculate very dense vortex tangles. Then, the fact that quantum vortices in superfluid helium are linear singularities is very important also from the point of view of applications for other systems, for example, for vacancies in solids or for cosmic strings.

In contrast to superfluid helium, where the nuclear size is on the order of interatomic distances, in a weak ideal Bose gas the core radius $a_0$ is much larger. Using equations (5) and (6), we can determine the structure of the vortex filament core. Near the central line of the vortex, the density of the superfluid component increases from zero and at a size of the order of $a_0$ becomes equal to the equilibrium value. In other words, quantum vortices in BEC, in contrast to He II (where they can be considered infinitely thin lines), are tubes with a size

$$a_0 = \frac{\hbar}{\sqrt{2mnU_0}} . \tag{7}$$

As a result, in numerical studies of QT in superfluid helium, such quantities as the characteristic size of the computational volume $D$, the typical distance between the lines and the nucleus radius $a_0$, are extremely separated in the space of scales. For example, in typical numerical studies, the computational volume is $\sim 1$ cm in size, the intervortex distance is on the order of $10^{-3}$– $10^{-4}$ cm, and the size of the nucleus $a_0$ in helium is about $10^{-8}$ cm, and so the total difference in scales is eight orders of magnitude. In the numerical simulation of BEC it is much smaller. For example, in the works of Kobayashi and Tsubota [23], the size of the computational volume was $D \approx 10^{-3}$ cm, while the size of the nucleus was $\sim 10^{-5}$ cm. Thus, these sizes differ only by two orders of magnitude. This quantitative difference leads to strong qualitative differences. Of course, one can choose such parameters in the GPE, so that the problem formally corresponds to the case of superfluid helium. However, this will require a record number of calculations (on a grid of approximately $100000^3$), which, certainly, is unrealistic.

The difference in the thicknesses of the vortex nuclei affects the laws of motion. For example, since the size of the nucleus in BEC is not very small compared to the inverse curvature of the line, then, unlike the case of He II, the velocity of the central line can be disproportionate to its curvature.

Another important difference between He II and BEC is that the latter is an essentially compressible gas. For small perturbations of density, linearised equations (5) and (6) yield a solution in the form of sound waves propagating with the speed of sound

$$c = \sqrt{\frac{U_0 n}{m}} . \tag{8}$$

The velocity of the linear elements of the vortex filament is not very small in comparison with the speed of sound. Indeed, it follows from expressions (7) and (8) that the speed of sound is $c \approx \kappa/a_0$ and the velocity of the elements of the line is $v_{\mathrm{line}} = \kappa/\delta$, where $\delta$ is the characteristic intervortex distance. Therefore, the Mach number $M = v_{\mathrm{line}}/c = a_0/\delta$ is not very small (in comparison with its value in the case of helium). This means that even large-scale movements of the filaments emit sound. In He II, this process takes place only for extremely small scales. In connection with this, the condition for the decay of the vortex structure in BEC, associated with the emission of sound, is much weaker. As a result, the kinetic energy related to the vortices decreases significantly, and the distribution of ordinary (nonsound) energy in the $k$-space (energy spectrum) changes.

Then, because of a 'thick' vortex core, the vortices in BEC lose a much larger part of their vortex energy during reconnection, emitting it in the form of phonons (due to a 'thick' vortex core and a low sound velocity). For example, recent calculations [33] showed that reconnection is accompanied by emission of sound waves with a wavelength on the order of the core radius $a_0$. In superfluid helium, energy losses during reconnection can be neglected.

In addition to the described differences between superfluid helium and BEC, there are such features as, for example, the inhomogeneity of the latter due to the presence of traps, and also a comparatively small number of atoms.

## 5. Conclusions

Thus, we have demonstrated that a quantitative difference in the parameters of quantum vortices leads to strong qualitative differences. In particular, compressibility is essential for BEC, vortices are not 'strings' and dynamics inside the core of vortex tubes is important. The compressibility property makes it impossible, for example, to compare the results for QT with the results of the Kolmogorov theory for classical turbulence. The fact that vortices in BEC are not string-like objects, does not allow one to apply the results obtained in the case of BEC to other systems with linear typological defects, in particular for dislocations or for space strings. The finite thickness of the vortex core also leads to a different dynamics of the vortex tubes in superfluid helium and in BEC.

It is often asserted that one can act absolutely formally and choose a very large value of $U_0$ in the GPE, such that the size of the vortex core $a_0$ is, on the contrary, small and the problem would correspond to the case of superfluid helium. However, in our opinion, this approach is incorrect. The fundamental reason is that in this case, in accordance with formulas (7) and (8), the interaction constant $U_0$ needs to tend to infinity. But the system, of course, ceases to be weakly interacting, and the Gross–Pitaevskii theory becomes inapplicable. Then, this procedure cannot be carried out in experimental studies, since the parameters of the system are specified by the properties of the gas. In numerical studies, as mentioned above, in this case a record amount of computation, exceeding the capabilities of any modern computers, will be required.

The caveats stated above show that any results (as numerical and experimental and so analytical) obtained for BEC cannot be automatically applied to the case of superfluid helium and vice versa. At the same time, separate research does not exhaust all scientific problems. It seems that the best option is to combine the two described approaches that complement each other.

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