Quantum Electronics **48** (8) 711 – 714 (2018)

PHOTON STATES OF THE ELECTROMAGNETIC FIELD https://doi.org/10.1070/QEL16698

What is a photon: structure and wave function

A.P. Kanavin, O.N. Krokhin

"For the rest of my life I will
reflect on what light is."
A. Einstein, 1917

Abstract. **Based on the model of a photon as a 'radiation oscillator' of a field moving in space with the speed of light and having a zero rest mass, we have constructed a wave function that describes the photon states of the electromagnetic field. It has been suggested that the entire space where the field is present splits into separate regions of the order of the radiation wavelength, within which the field is coherent and the total field energy is equal to the photon energy.**

Keywords: *Maxwell's equations, photon, wave function.*

The concept of a photon – "a quantum of the electromagnetic field" – was introduced by M. Planck and A. Einstein* at the beginning of the 20 century [2, 3]. Recently, there has been a great interest in studying the quantum states of the electromagnetic field (photon states) in order to understand how an elementary particle of this field is constructed [4]. A critical review devoted to various aspects of the concept of 'photon' is presented in [5]. St. Weinberg defines the concept of 'an elementary particle' as a bundle of corresponding fields [6].

In the modern interpretation of quantum electrodynamics, the concept of an elementary particle-photon is introduced somewhat formally through the production and annihilation of quanta of the electromagnetic field in the asymptotic approximation in the entire space and time [7]. At the same time, the question of the structure of elementary particles does not arise, although the quanta of electromagnetic radiation – photons – have relatively large dimensions ($\sim$1 μm – of the order of the wavelength) in the optical spectral range and can be 'probed' by modern methods of investigation.

We will try to construct a wave function describing the photon itself, i.e., in essence, the structure of the electromagnetic field of the photon. The starting point for this is the model of a photon as a 'radiation oscillator' of a field moving in space at the speed of light and having a rest mass equal to zero.

* In his letter to M. Besso, on 12 December 1951 A. Einstein wrote, "All these fifty years of pondering have not brought me any closer to answering the question, What are light quanta? Nowadays every Tom, Dick and Harry thinks he knows it, he is mistaken" [1].

A.P. Kanavin, O.N. Krokhin P.N. Lebedev Physical Institute, Russian Academy of Sciences, Leninsky prosp. 53, 119991 Moscow, Russia; National Research Nuclear University 'MEPhI', Kashirskoe sh. 31, 115409 Moscow, Russia; e-mail: kanavin@sci.lebedev.ru

Received 11 April 2018
Kvantovaya Elektronika **48** (8) 711 – 714 (2018)
Translated by I.A. Ulitkin

Thus, we will assume that the photon's wave function consists of a product of two functions: a function describing its free motion with momentum $\boldsymbol{p}$, i.e., a de Broglie wave, and a function describing its structure. We adhere to the point of view that this problem is analogous to the problem of the wave function of a massive particle that has an internal structure, for example, a hydrogen atom that freely moves in space.

We choose a coordinate system. Let the z axis be directed along the direction of the photon motion. The x axis is directed in the direction of the electric field vector E (linear polarisation of the electromagnetic wave is considered), and the y axis is perpendicular to the xz plane. In this plane, at the origin, the monochromatic wave vector E performs harmonic oscillations with a cyclic frequency ω like a linear oscillator. Let us try to reduce the problem of a photon, sometimes called the 'radiation field oscillator', to the problem of the quantum states of a mechanical linear oscillator [7].

We will try to build on this basis a quantum approach to the solution of the problem of the photon structure. To do this, we consider a classical oscillator performing harmonic oscillations along the x axis. The problem of the quantum behaviour of the oscillator was first considered by W. Heisenberg in 1925, even before the discovery of the Schrödinger equation [8]. Heisenberg applied the matrix method he developed, relying on the commutation rules for the canonically conjugate quantities of the operators of the coordinate x and momentum $\hat{p} = -\mathrm{i}\hbar\partial/\partial x$.

For the 'radiation oscillator' the role of the coordinate is played by the electric field strength $E(t, x)$, which oscillates in time with a frequency equal to that of the electromagnetic wave. Thus, we can use the relation [8]:

$$\frac{\partial E}{\partial t} + \mathrm{i}\omega E = 0. \qquad (1)$$

In going over to a quantum analysis, the quantities E and $\partial E/\partial t$ should be replaced by noncommuting operators $\hat{E}$ and $\partial\hat{E}/\partial t$.

Note that in the classical case, at the point where $\partial E/\partial t = 0$, the value of E reaches a maximum value equal to the amplitude of the oscillations $E_0(x, 0)$, which, in turn, defines the time-averaged field energy density in the wave and the energy flux density (the latter taking into account the equality of electric and magnetic fields are equal to $E_0^2/8\pi$ and $cE_0^2/8\pi$, respectively). After E becomes equal to E_0, it will decrease with time like the coordinate of a mechanical oscillator. The behaviour of E and the coordinates of the mechanical oscillator are very similar: both quantities perform harmonic oscillations, and the energy characteristics in both cases are proportional to the squares of the corresponding quantities.

In the case of quantum description, the system in a stationary state, i.e., in a state with a given energy, is described by the wave function, which is the product of two functions that depend only on the coordinate and only on the time:

$$\psi(x,t) = \psi(t)\psi(x), \tag{2}$$

where $\psi(t) = \exp(-\mathrm{i}\omega t)$, and $\psi(x)$ determines the structure of the object, i.e., the wave field corresponding to the standing wave. To determine $\psi(x, t)$, it is necessary to find a solution to the 'boundary value problem'.

Equation (1), written for operators, has the form

$$\frac{\partial \hat{E}}{\partial t} + \mathrm{i}\omega\hat{E} = 0. \tag{3}$$

It coincides with the equation for the operator $\hat{x}$, i.e. the operator of the Heisenberg linear oscillator coordinate, with the only difference being that the mechanical oscillator oscillates with respect to point with $x = 0$, to which the elastic force returns it, and the field E does not have such a point – it is not local and oscillates in the entire volume, where this field is present.

Since equation (3) coincides with the equation

$$\frac{\partial \hat{x}}{\partial t} + \mathrm{i}\omega\hat{x} = 0, \tag{4}$$

which describes the quantum behaviour of a mechanical oscillator (here we have chosen the sign '+' in accordance with formula (23.11) in [8]), their solutions are analogous. Thus, our problem reduces to replacing $\hat{E}$ by $\hat{x}$, and, accordingly, the electric field strength operator $\hat{E}$ does not commute with the rate of its change.

In our coordinate system at a given time (for example, at $t = 0$), the electric field directed along the x axis has the same value in the xy plane. Therefore, as a point in which an oscillating electric field is present, we can choose a point with an arbitrary coordinate (x, y). This corresponds to a wave of the electromagnetic field (or probability wave) when all the parameters on the phase surface are the same (for example, a plane wave). For simplicity, we set $x = y = 0$ for this point.

It can be assumed that is the photon is located near this point and the next task is to define the structure of this photon (the determination of the spatial dependence of the electric field in this region).

Here one important remark should be made. It refers to the question of the coherence of the electromagnetic field. Since we introduced the field frequency ω specified by the source, the question arises of the spatiotemporal coherence of a monochromatic field. Namely, if we have an ideally monochromatic field, then it is the same in the entire space and time, which are formally unlimited. However, this is impossible, since fluctuations and changes in its amplitude, phase (i.e., frequency), and spatial properties take place during the emission of a wave. These properties are determined by the so-called spatiotemporal correlation function of the electromagnetic field of the wave. As a result, the entire space where the field is present breaks up into separate regions on the order of the radiation wavelength, within which the field is coherent, and its total energy is equal to the photon energy.

In this connection, reference should be made to the work of Ya.B. Zel'dovich, in which the total number N of quanta of the electromagnetic field is determined in terms of the correlation integral [9]:

$$N = \frac{1}{2c\hbar}\iint \frac{\boldsymbol{E}(\boldsymbol{r}_1)\boldsymbol{E}(\boldsymbol{r}_2) + \boldsymbol{H}(\boldsymbol{r}_1)\boldsymbol{H}(\boldsymbol{r}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}\frac{\mathrm{d}\boldsymbol{r}_1\mathrm{d}\boldsymbol{r}_2}{(2\pi)^3}, \tag{5}$$

where $\boldsymbol{H}$ is the intensity of the magnetic field; $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the radius vectors of two different points in space; and the multiplier $1/|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2$ determines the correlation of the field at these points. Here, in fact, Huygens' idea is used that every point of the wave front radiates spherical waves and, consequently, radiation from a point with a radius vector $\boldsymbol{r}_1$reaches a point with a radius vector $\boldsymbol{r}_2$ weakened by $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ times. This ensures the finiteness of the volume of integration and is actually equivalent to the introduction of a spatial correlation function.

The 'speed' $\partial E/\partial t$, as follows from Maxwell's equations, can be expressed in terms of the spatial derivative $c^{-1}(\partial E/\partial t) = -\partial E/\partial x$ and, thus, we obtain

$$c\frac{\partial E}{\partial x} - \mathrm{i}\omega E = 0. \tag{6}$$

Multiplying this equation by $\mathrm{i}\hbar/c$, we obtain

$$\mathrm{i}\hbar\frac{\partial E}{\partial x} + \frac{\hbar\omega}{c}E = 0, \tag{7}$$

After that, introducing the momentum operator $\hat{p}_x = -\mathrm{i}\hbar\partial/\partial x$, we can write

$$\hat{p}_x E = \frac{\hbar\omega}{c}E. \tag{8}$$

Equation (8) is a relation that determines the eigenvalues of the momentum operator and the boundary conditions for $\partial E/\partial t$, namely, $\partial E/\partial t = 0$ when E reaches the maximum value, i.e., $E_0 = \sqrt{8\pi S}/c$ (S is the radiation flux density). Then,

$$E = \pm E_0 \text{ at } \partial E/\partial t = 0, \qquad E = 0 \text{ at } \partial E/\partial t \to \max|\partial E/\partial t|. \tag{9}$$

The first of these conditions can be considered as a boundary condition for $E(x)$.

Here an important question arises, connected with the dualism of quantum 'wave–particle' physics, which is considered, in particular, in [9]. Namely, how much is the electromagnetic field coherent on the phase plane xy? In our case, the electric field is polarised along the x axis and depends only on x. The characteristic coherence length is determined by the wavelength $\lambda = 2\pi c/\omega$; therefore, equation (8) is valid only at distances $x \leqslant \lambda$. Thus, this equation can be written in the form

$$\frac{\partial E}{\partial x} = -\frac{2\pi x}{\lambda^2}E. \tag{10}$$

The factor x/λ reflects the fact that for $x = 1$ the field energy enclosed in the volume λ^3 is equal to the quantum energy $\hbar\omega$. It follows from this that it is equivalent to the wave function of a linear mechanical oscillator (see [8], §23). Since the wave electric field can be considered as a wave function of a photon in quantum electrodynamics (up to a normalising factor), the solutions given in [8] are, in essence, a solution for the electromagnetic quantum particle of the 'radiation' oscillator, i.e. a photon.

Namely, the ground state with $n = 0$ is zero field oscillations, and the first excited state with $n = 1$ corresponds to one photon. Essentially, the solution of equation (10) is 'standing' waves (states) in a parabolic potential proportional to E^2 (Fig. 1), where E is the current value of the electric field strength of the wave. The solutions normalised to unity for $E(x) = \psi_0(x)$ are Hermite polynomials and give the energy density distribution of the electromagnetic field along the transverse coordinate x.

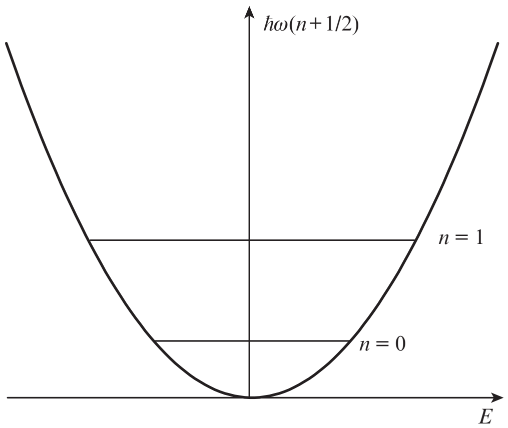

Figure 1. Photon energy for zero oscillations ($n = 0$) and a single-photon state ($n = 1$).

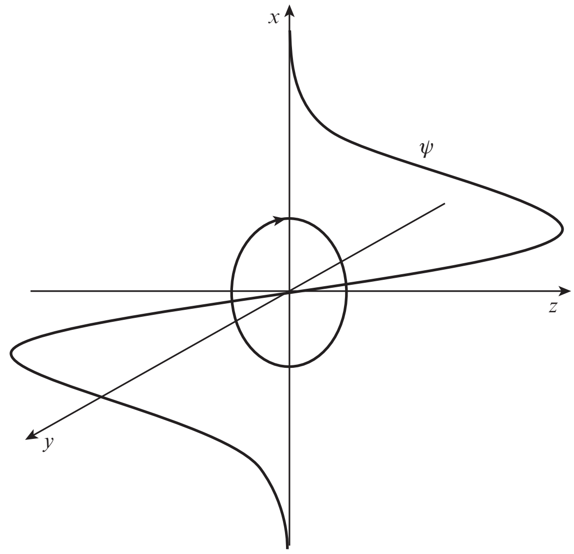

Figure 2. Wave function of a circularly polarized photon propagating along the z axis. The field is described by the first Hermite polynomial ($n = 1$).

Thus, the total wave function of the photon, the state corresponding to the quantum number $n = 1$, is the product of the wave function describing the photon propagation in free space and the wave function in the form of the first Hermite polynomial describing the photon structure (Fig. 2).

Of course, there remains a fundamental question: why is an elementary particle, i.e. a photon, stable? It would seem that the presence of a dependence of ψ on the transverse coordinate x should lead to diffraction and the 'decay' of the photon over time. However, it seems to us that any violation of the solution found should be accompanied by a 'rearrangement' of the wave function, which leads to an increase in the energy of the given state, which is impossible. Formally, this is expressed in the fact that the solution found is stationary and $\hbar\omega = \text{const}$. The physical reason for this is the lack of field sources.

Let us pay attention to the following circumstance. The energy density of an electromagnetic field with its linear polarisation oscillates in space and time in proportion to E_x^2, but for a photon the energy is a constant equal to hw. This discrepancy is eliminated we assume that the photon has circular polarisation with the field components E_x and E_y, displaced in phase from each other by $\pi/2$. Given that the expression for E_y is the same as the expression for E_x (with x replaced by y), for $|E_\perp|^2$ we have a value that is constant in time and space. This removes the above contradiction and leads to the fact that the photon must have a momentum $M \sim |\boldsymbol{r} \times \boldsymbol{p}|$, which, taking into account the fact that $|\boldsymbol{r}| \sim \lambda$, and $|\boldsymbol{p}| = \hbar\omega/c$, is equal to $\hbar$ (see [10]).

Here one important remark should be made. The total energy of a mechanical oscillator is equal to the sum of the kinetic and potential energies, and is constant in the stationary case. In the process of oscillations, the kinetic and potential energies convert into each other. For the 'radiation oscillator' only in the case of circular polarisation of the wave, the total energy is a constant value, and this means that the sum of the vibrational energies along the x and y axes is an integral of motion analogous to the above-mentioned case of a harmonic oscillator. This provision, in essence, reveals specifically the statement about the analogy between the 'radiation oscillator' and the mechanical one. This also means that a photon must be circularly polarised and have a moment of a momentum equal to $\hbar$.

In conclusion, it should be emphasised that the transition from the classical description of the electromagnetic field (Maxwell's equations) to the quantum representation is closely related to the concept of coherence, as was noted earlier by Ya.B. Zel'dovich, who in introduced an expression for the number of field quanta. In this case, the region of space where the electromagnetic wave is present as if decays into 'domains', the number of which corresponds to the number of photons, i.e., the field in one 'domain' is equivalent to the field of one photon, and the linear size of this 'domain' is equal to the wavelength.

Acknowledgements. The authors thank A.V. Vinogradov and I.G. Zubarev for fruitful and stimulating discussions.

References

1. Frankfurt U.I. (Comp.) *Einshteinovskii sbornik 1977* (Einstein's Collected Papers, 1977) (Moscow: Nauka, 1980) p. 41.
2. Planck M. *Verh. Deutsch. Phys. Ges.*, **2**, 202, 237 (1900); Planck M. *Teoriya teplovogo izlucheniya* (Theory of Thermal radiation) (Moscow: ONTI, 1935); Planck M. *Izbrannye trudy* (Selected Works) (Moscow: Nauka, 1975).
3. Einstein A. *Ann. Phys.*, **17**, 132 (1905); **20**, 199 (1906); *Phys. Zeits.*, **10**, 185, 817 (1909); Einstein A. *Sobranie nauchnykh trudov. Tom III. Raboty po kineticheskoi teorii,*

teorii izlucheniya i osnovam kvantovoi mekhaniki 1901–1955 (Collected Scientific Works. Vol. III. Works on the Kinetic Theory, Radiation Theory and the Fundamentals of Quantum Mechanics, 1901–1955) (Moscow: Nauka, 1965–1967).
4. Greenstein J., Zajons A. *The Quantum Challenge. Modern Research of the Foundations of Quantum Mechanics* (Sudbury, Massachusetts: Jones and Barlett Publishers, 2006; Dolgoprudny: ID Intellekt, 2008).
5. Klyshko D.N. *Sov. Phys. Usp*., **37**, 1097 (1994) [*Usp. Fiz. Nauk*, **164**, 1187 (1994)].
6. Weinberg St. *Sci. Am*., **281** (6), 68 (1999); Weinberg St. *V zashchity nauki* (In Defence of Science) (Moscow: Nauka, 2012) No. 11, p. 71.
7. Akhiezer A.I., Berestetsky V.B. *Quantum Electrodynamics* (New York: Interscience Publishers, 1965; Moscow: Nauka, 1981).
8. Landau L.D., Lifshitz E.M. *Quantum Mechanics* (Oxford: Pergamon Press, 1965; Moscow: Fizmatgiz, 1963).
9. Zel'dovich Ya.B. *Dokl. Akad. Nauk SSSR*, **163** (6), 1359 (1965); Zel'dovich Ya.B., Raizer Yu.P. *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena* (New York: Dover Publications; 2002; Moscow: Fizmatlit, 2008).
10. Akhmanov S.A., Nikitin S.Yu. *Physical Optics* (Oxford: Clarendon Press, 1997; Moscow: Izd-vo MGU, Nauka, 2004).