Quantum Electronics **48** (8) 720–727 (2018)

https://doi.org/10.1070/QEL16709

Self-action of a wave field in one-dimensional system of weakly coupled active optical waveguides

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Abstract. **The dynamics of self-action of a wave field in an array of weakly coupled active optical waveguides is studied analytically and numerically. It is shown that the introduction of weak gain in each of the waveguides allows significant reduction of radiation losses and efficient capture of all radiation in a single optical waveguide. The possibility of controlling the radiation localisation in the desired waveguide at the expense of varying the injection angle of the wave field into the considered system is demonstrated. The dependences of the characteristic parameters of the wave field are determined.**

Keywords: *weakly coupled active optical waveguides, self-action, localisation of radiation in a waveguide.*

1. Introduction

Considerable technological progress in fabricating microstructured waveguide systems with given diffraction and dispersion properties has led to the active development of the nonlinear science research, devoted to nonlinear wave processes in spatially periodic media – arrays of weakly coupled optical waveguides [1–3]. Besides the purely fundamental interest, the performed studies have practical aims, namely, the supercontinuum generation [4] and reduction of laser pulse duration [5, 6], the control of wave field structure [7, 8], and the light bullet formation [9, 10].

A distinctive feature of such 'discrete' media is the existence of a stable distribution of the wave field localised on a period of the structure in a wide range of amplitudes – the discrete soliton [8]. Alongside with this fact, it was experimentally shown in Ref. [11] that when the pulse power exceeds a critical value, the initially wide distribution of the wave field as a result of the discrete collapse development localises in regions, whose size is comparable with the period of the lattice of equidistantly arranged optical waveguides. In Ref. [12], this critical power was determined basing on the developed variational approximation. However, in the process of localising the radiation in a region, the size of which is approximately equal to the period of the medium inhomogeneity, the 'radiation losses' of the wave field and the corresponding decrease of the power in the narrow central part of the beam are observed.

In the present paper, we consider the dynamics of self-action of the wave field in a one-dimensional system of weakly coupled active optical waveguides. It is shown that the introduction of weak gain in each of the optical waveguides allows considerable reduction of radiation losses and provides the efficient capture of all radiation in one optical waveguide. This is achieved due to an adiabatic decrease in the transverse size of the beam with an almost plane phase front, which provides the smoothness of the radiation injection into one waveguide. In this case, the discrete collapse occurs at the final stage. Besides that, we demonstrate the possibility of controlling the localisation of radiation in a desired waveguide by varying the angle of the wave field injection into the considered system.

To describe the self-action dynamics of a wave field the discrete nonlinear Schrödinger equation [1, 8, 12–14] is used with the additional term responsible for the gain of the wave field. The use of variational approximation (Section 2) allows qualitative analysis of dynamics in the system and estimation of the basic parameters of the wave field Gaussian distribution (the beam width, the phase front curvature, the coordinate of the intensity maximum of the wave beam). In Section 3, we study the dynamics of the wave field injected along the axis of the considered system. Efficient capture of all radiation in the central optical waveguide is demonstrated. The estimate of the length of this capture is obtained. In Section 4, we analyse the dynamics of wave beams injected at a certain angle to the considered system. The possibility of controlling the radiation localisation in the required waveguide by varying the radiation injection angle is shown. The analytic dependence of this shift on the parameters of the problem is found.

2. Formulation of the variational problem

Consider the self-action of a wave field, injected into the spatially inhomogeneous medium, formed by an array of equidistantly arranged single-mode optical waveguides with the permittivity

$$\varepsilon \approx \varepsilon_0 + \varepsilon_1 \sum_n s(\boldsymbol{r}_\perp - \boldsymbol{l}n). \quad (1)$$

Here $\boldsymbol{l}$ is the lattice period of optical waveguides; $s(\boldsymbol{r}_\perp) = \mathrm{step}(|\boldsymbol{r}| - R)$ is a narrowly localised function with the characteristic scale $R \ll l$; n is the optical waveguide number; and R is the waveguide radius.

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Received 19 April 2018
Kvantovaya Elektronika **48** (8) 720–727 (2018)
Translated by V.L. Derbov

Let us assume that each waveguide is single-mode. In this case, the propagation of the wave field $\mathcal{E}$ in the considered system can be approximately described as a superposition of modes, localised in each of the optical waveguides:

$$\mathcal{E}(\boldsymbol{r}_{\perp},z)=\sum_n \Psi_n(z)f(\boldsymbol{r}-\boldsymbol{l}n), \tag{2}$$

where $f(\boldsymbol{r})$ is the function that determines the fixed structure of the lowest mode of the waveguide; and Ψ_n is the complex amplitude of the field in the nth waveguide. The analysis will be carried out basing on the standard theoretical model [1, 8, 14], in which it is assumed that the fundamental guided modes of the optical waveguides, oriented parallel to the z axis, are weakly coupled to each other. Let us assume that the evolution of the field envelope Ψ_n in the nth optical waveguide is determined in the process of wave field propagation along the z axis by the following factors: the Kerr nonlinearity in each optical waveguide, the gain in the active medium, and the coupling with the adjacent waveguides due to a weak overlap of their guided modes.

For the medium consisting of an unlimited number of equidistantly arranged optical waveguides, the discrete nonlinear Schrödinger equation in dimensionless variables has the form

$$\mathrm{i}\frac{\partial \Psi_n}{\partial z}+\Psi_{n+1}+\Psi_{n-1}+|\Psi_n|^2\Psi_n=\mathrm{i}\gamma\Psi_n. \tag{3}$$

Here the parameter γ characterises the linear gain in the active medium.

To investigate the dynamics of the systems we use the variation approximation that allows classification of characteristic regimes of self-action and determination of the appropriate critical parameters. The approximation consists in minimisation of the action functional using the probe function Ψ that depends on a few parameters, which are functions of the evolution coordinate z. Technically the method is based on the transition to the reduced Lagrangian $\bar{\mathcal{L}}$, obtained by integrating the initial Lagrangian $\mathcal{L}$ with the substituted probe functions over the transverse coordinates.

In this Section we develop the variational approach in application to the studied problem of self-action of a wave field in the nonconservative discrete system (3), reducing the problem to a closed system of ordinary differential equations for the characteristic integral parameters of the Gaussian-shaped distribution of the wave field (the wave packet width, the phase front curvature, etc.). Although the situation becomes more complex in the case of an active medium because of non-Hamiltonian properties of the system, the variational problem still can be formulated for the system (3). In this case, alongside with the Lagrangian of the conservative part of the system, it is also necessary to define the dissipative function of the system.

The Lagrange function, corresponding to Eqn (3) in the conservative case ($\gamma=0$), has the form [12, 14–16]

$$\mathcal{L}=\sum_{n=-\infty}^{+\infty}\left[\frac{\mathrm{i}}{2}\left(\Psi_n^*\frac{\partial\Psi_n}{\partial z}-\Psi_n\frac{\partial\Psi_n^*}{\partial z}\right)\right.$$
$$\left.+\frac{1}{2}|\Psi_n|^4+\Psi_{n+1}\Psi_n^*+\Psi_{n+1}^*\Psi_n\right]. \tag{4}$$

Using the Poisson summation formula for the function $F(x)$ of a continuous argument

$$\sum_{n=-\infty}^{+\infty}F(n)=\int_{-\infty}^{+\infty}F(x)\sum_{n=-\infty}^{+\infty}\exp(2\pi\mathrm{i}nx)\,\mathrm{d}x,$$

let us rewrite Lagrangian (4) in the form

$$\mathcal{L}=\sum_{n=-\infty}^{+\infty}\exp(2\pi\mathrm{i}nx)\int_{-\infty}^{+\infty}\left[\frac{\mathrm{i}}{2}\Psi^*(z,x)\frac{\partial\Psi(z,x)}{\partial z}\right.$$
$$\left.+\Psi(z,x+1)\Psi^*(z,x)+\text{c.c.}+\frac{1}{2}|\Psi|^4\right]\mathrm{d}x. \tag{5}$$

This allows the use of a single function $\Psi(z,x)$, depending of the continuous variable x, instead of the infinite ordered set of complex field amplitudes $\Psi_n(z)$ in each of the waveguides.

To describe approximately the propagation of a localised wave packet, injected into the one-dimensional periodic system of optical waveguides, let us consider the Gaussian amplitude distribution

$$\Psi=\sqrt{\frac{P}{\sqrt{\pi}\,a}}$$
$$\times\exp\left[-\frac{(x-x_0)^2}{2a^2}+\mathrm{i}\beta(x-x_0)^2+\mathrm{i}\sigma(x-x_0)+\mathrm{i}\theta\right] \tag{6}$$

as an approximation for the function $\Psi(z,x)$. Here

$$P=\int_{-\infty}^{+\infty}|\Psi|^2\mathrm{d}x$$

is the wave field power; $a(z)$, $\beta(z)$, and $\theta(z)$ are the wave beam width, the phase front curvature, and the phase of the field on the system axis, respectively; x_0 is the coordinate of the intensity maximum of the wave beam; and σ is the transverse wavenumber. Substituting Eqn (6) into Eqn (5) and integrating the obtained expression over the continuous variable x, we arrive at a functional series. For wide beams with $a(z)\gg 1/\pi$ the coefficients of this series exponentially decrease with increasing n (see, e.g., [12, 15]). Thus, even for the wave fields with the characteristic transverse dimension $a(z)$, comparable with the scale of the medium spatial inhomogeneity, we will restrict ourselves to considering only the terms with $n=0$, when using Eqn (5) for the description of self-action processes. Finally, for the reduced Lagrange function $\bar{\mathcal{L}}$ of the conservative part ($\gamma=0$) of the considered system (3), we obtain the expression:

$$\bar{\mathcal{L}}=\frac{Pa^2}{2}\frac{\mathrm{d}\beta}{\mathrm{d}z}+P\cos\sigma\frac{\mathrm{d}\theta}{\mathrm{d}z}-\frac{P^2}{\sqrt{8\pi}\,a}$$
$$-2P\exp\left(-\frac{1}{4a^2}-\beta^2a^2\right)\cos\sigma-P\sigma\frac{\mathrm{d}x_0}{\mathrm{d}z}. \tag{7}$$

The evolution of the parameters $a_j=\{P,a,\beta,\sigma,x_0,\theta\}$ along the path of the wave beam propagation is determined by the Euler equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial\bar{\mathcal{L}}}{\partial a_j'}-\frac{\partial\bar{\mathcal{L}}}{\partial a_j}=0,\quad a_j'=\frac{\mathrm{d}a_j}{\mathrm{d}z}. \tag{8}$$

The generalisation of the Euler equations for the determination of the parameters of the variational function (6) in the nonconservative case consists in considering the contribution of the dissipative part. For Eqn (3) the variation of the dissipative function is

$$\delta Q = \mathrm{i}\gamma \int_{-\infty}^{+\infty} (\Psi \delta \Psi^* - \Psi^* \delta \Psi)\,\mathrm{d}x\,. \qquad (9)$$

Therefore, the Euler equations for the parameters of the wave packet (6) in the considered nonconservative case have the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial \bar{\mathcal{L}}}{\partial a_j'} - \frac{\partial \bar{\mathcal{L}}}{\partial a_j} = \mathrm{i}\gamma \int_{-\infty}^{+\infty} \left(\Psi \frac{\partial \Psi^*}{\partial a_j} - \mathrm{c.c.} \right) \mathrm{d}x\,. \qquad (10)$$

Performing appropriate calculations for the wave beam (6), we find that $\mathrm{d}\sigma/\mathrm{d}z = 0$ and $\sigma = \sigma_0$. The rest equations that determine the evolution of the parameters of the wave beam along the propagation trace have the form

$$\frac{\mathrm{d}P}{\mathrm{d}z} = 2\gamma P\,, \qquad (11\mathrm{a})$$

$$\frac{\mathrm{d}a}{\mathrm{d}z} = 4\beta a d \cos\sigma_0\,, \qquad (11\mathrm{b})$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}z} = \frac{\cos\sigma_0}{a^2}\left(\frac{1}{a^2} - 4\beta^2 a^2\right) d - \frac{P}{\sqrt{8\pi}\,a^3}\,, \qquad (11\mathrm{c})$$

$$\frac{\mathrm{d}x_0}{\mathrm{d}z} = 2d \sin\sigma_0\,. \qquad (11\mathrm{d})$$

Here the factor

$$d = \exp\left(-\frac{1}{4a^2} - \beta^2 a^2\right) \qquad (12)$$

reflects the specificity of the discrete problem. In the continuous case ($a \gg 1, \beta a \ll 1$) it is close to unity. In the opposite limit case (both a and βa are of the order of unity) the factor d is exponentially small, i.e., the discreteness leads to the weakening of linear diffraction of the wave field. The presence of gain in the medium ($\gamma > 0$) causes an exponential growth of the wave beam power (11a):

$$P = P_0 \exp(2\gamma z)\,, \qquad (13)$$

where P_0 is the initial power.

The derived system of Eqns (11b)–(11d) is a generalisation of the corresponding equations of the conservative problem [12, 17] for the case of active spatially inhomogeneous medium. The spatial evolution of the wave beam and the behaviour of the parameter x_0 are described by the same equations (11b)–(11d) as in Ref. [17], with the only difference that the power of the wave beam grows along the propagation path in accordance with the exponential law (13). Therefore, at the qualitative level we should expect here a similar picture of the wave field self-action. Below we dwell on considering in more detail the new effects determined by the medium gain.

3. Self-localisation of the wave beam injected along the system axis

Within the frameworks of the used approach, it is seen that the structure changes of the wave field, described by Eqns (11a)–(11c) occur independent of the behaviour of the intensity maximum coordinate x_0 (11d) of the wave beam. The trajectory of motion $x_0(z)$ is determined by the separate equation (11d). It is important to note that the deviation of the trajectory from a straight line in a continuous medium reflects the specificity of the spatially inhomogeneous problem. In the case of nonlinear dynamics, the transverse wavenumber is preserved ($\sigma = \sigma_0 = \mathrm{const}$), characterising the initial angle between the direction of the beam propagation and the axis of the system. In this connection, we first analyse the specific self-action features for a wave beam, injected at the input of the one-dimensional lattice of weakly coupled active optical waveguides in the case $\sigma_0 = 0$. As follows from Eqn (11d), the coordinate of the intensity maximum of the wave beam will not change: $x_0 = 0$.

The main feature of radiation self-action in the conservative case is related to the 'collapse' of the wave field (capture in one waveguide). The discreteness of the medium leads to the weakening of diffraction for the narrow wave beams compared to the lattice period. Therefore, even in the conservative case, the collapse of one-dimensional distribution of the wave field with the power, exceeding the critical power of self-focusing in the discrete medium becomes possible [11, 12, 17].

In contrast to the discrete conservative medium, in the active medium the capture of radiation in one optical waveguide occurs smoothly due to the adiabatic reconstruction of the soliton distribution under the field amplification. The soliton distribution of the wave field corresponds to a stationary point of Eqns (11b) and (11c). At the stationary point the curvature $\beta = 0$ (i.e., the wave beam has a plane phase front), and the width a is determined by the equation

$$P = P_{\mathrm{sol}}(a) \equiv \frac{\sqrt{8\pi}}{a} \exp\left(-\frac{1}{4a^2}\right). \qquad (14)$$

Note that this dependence of the power on the width has an extremum $P_{\max} = 4\sqrt{\pi/\mathrm{e}}$ at $a_{\max} = 1/\sqrt{2}$, corresponding to the limiting width of the soliton in the discrete system.

Assuming the invariance of the soliton shape of the wave beams, relation (14) between the characteristic width a of the field distribution and the power P of the wave beam is conserved over the entire propagation path. In the case of wide spatial solitons ($a \gg 1$), the condition for the conservation of the soliton shape will consist in slow variation of the beam power (13) at the diffraction length z_{d}:

$$z_{\mathrm{d}} \approx a^2 \approx \frac{8\pi}{P^2} \ll \frac{1}{\gamma}\,. \qquad (15)$$

The exponential growth of the radiation power P (13) will lead to an adiabatic decrease in the spatial soliton size. From Eqn (14), we find the following law of the beam width decrease:

$$a \approx \frac{\sqrt{8\pi}}{P} - \frac{P}{8\sqrt{2\pi}} \approx a_0 \exp(-2\gamma z) \;\; \text{for} \;\; P \ll \sqrt{8\pi}\,, \qquad (16)$$

where $a_0 \approx \sqrt{8\pi}/P_0$ is the initial soliton width. It is worth noting that the analogous possibility of an adiabatic decrease in the soliton duration in active optical media according to the exponential law was discussed in Ref. [18].

The process of the wave beam narrowing is limited by the discreteness of the medium, if the width of the wave field distribution becomes on the order of the lattice period: $a \approx 1$. Then the length L_h of the wave field capture by the central optical waveguide is easily found from Eqn (14) using the substitution $a_{fin} = 1$:

$$L_h \approx \frac{1}{2\gamma} \ln \frac{P_{cr}}{P_0} \quad (P_{cr} = \sqrt{8\pi}\, e^{-1/4}). \tag{17}$$

Note that the determined value of the threshold power of radiation capture for active systems $P_{cr} \approx 3.9$ is close to the critical power of the wave beam collapse in a one-dimensional discrete system $P_c \approx 3.7$, found for the conservative case [12, 17].

Let us proceed to the results of numerical simulation. Figures 1a and 1b, present the evolution of the width $a(z)$ of the spatial soliton (6), (14) with the initial size $a_0 = 25$, injected into the one-dimensional lattice of equidistantly arranged active optical waveguides for the gains $\gamma = 10^{-4}$ and 10^{-3}. The solid curve shows the evolution of the root-mean-square beam size

$$\langle a \rangle = \frac{1}{\sqrt{P(z)}} \sqrt{\sum_{n=-\infty}^{+\infty} n^2 |\Psi(z,n)|^2}\,, \tag{18}$$

calculated basing on the numerical simulation of Eqsn (3) and the dashed curve shows the dynamics of the beam width, calculated by means of Eqns (11). As follows from the results of numerical simulation, presented in Figs 1a and 1c, the data, obtained from the solution of the initial system of equations (3) agree well with the results of the qualitative analysis of the problem based on the variational approach (11). With an increase in gain γ, the regime of adiabatic decrease in the beam size is disturbed insignificantly. In Figs 1a and 1c one can see a characteristic step in the dependence of the beam size on z (at $z \gtrsim 15000$ for $\gamma = 10^{-4}$ and at $z \gtrsim 1500$ for $\gamma = 10^{-3}$), which is related to the discrete collapse. Therefore, the length of radiation capture in the central optical waveguide is smaller than the estimate (17), which is due to faster collapse development.

Figures 1b and 1d show the dependence of the phase front curvature β on the beam width a for two gains γ. It is seen that the wave beam conserves the plane phase front ($\beta \approx 0$) up to the beam size $a \approx 2$; this also confirms the assumption of adiabatic decrease in the beam size during radiation amplification. However, with a further decrease in the beam width a, the phase front becomes nonplanar; the curvature $\beta \neq 0$ is nonzero and grows along the beam propagation trace.

Thus, we can select two stages of the system evolution. At the initial stage, while the radiation power is below the critical one ($P(z) < P_{cr}$) the adiabatic decrease in the beam size occurs in the process of amplification in the active medium, and upon the achievement of the critical power the one-dimensional collapse occurs, and the wave beam becomes captured in one channel.

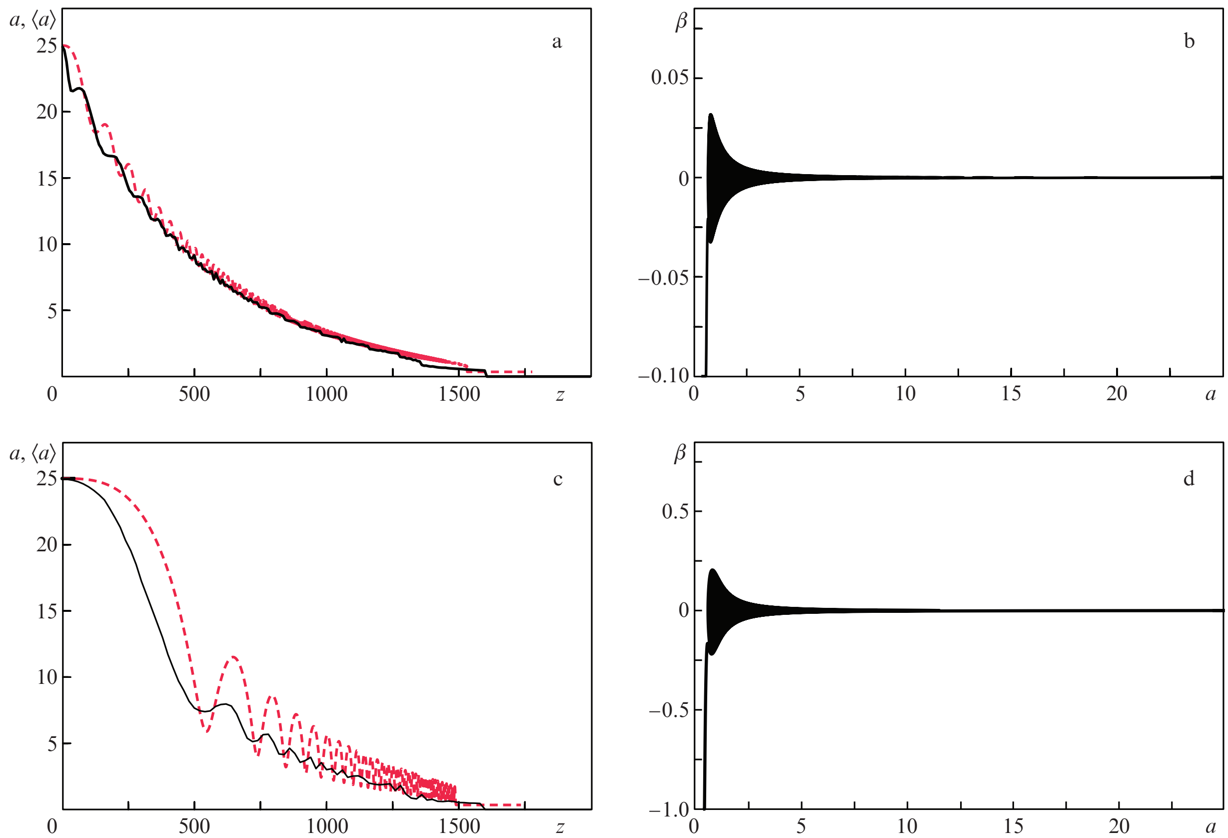

Figure 1. (a, c) Evolution of the spatial soliton size a (dashed lines) and $\langle a \rangle$ (solid lines), as well as (b, d) dependence of the phase front curvature β on the beam width a for the gains γ = (a, b) 10^{-4} and (c, d) 10^{-3}; $a_0 = 25$.

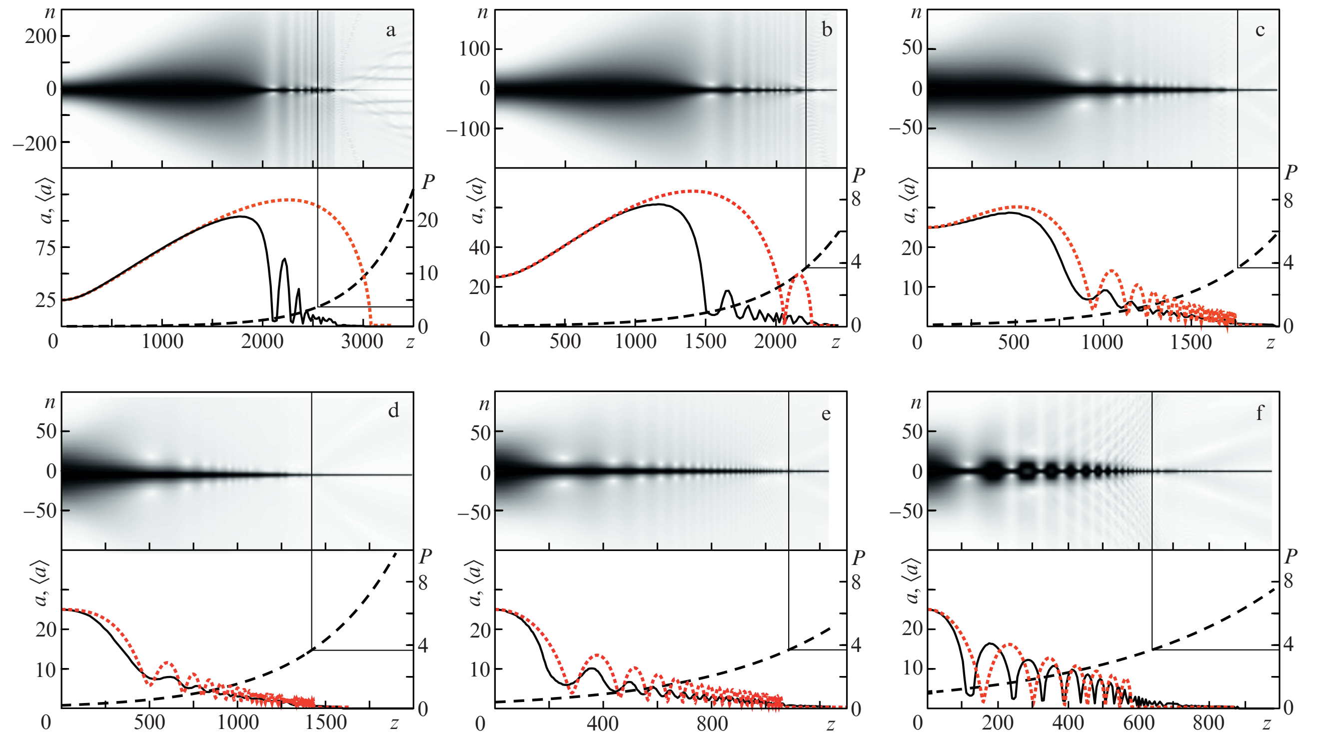

Figure 2. Dynamics of the wave beam amplitude $|\Psi(z,n)|$ and the dependences of the power P (dashed lines), beam size a (dotted lines) and $\langle a\rangle$ (solid lines) on z at the initial powers $P_0 =$ (a) $0.1P_{sol}$, (b) $0.2P_{sol}$, (c) $0.5P_{sol}$, (d) P_{sol}, (e) $2P_{sol}$ and (f) $5P_{sol}$; $\gamma = 10^{-3}$, $a_0 = 25$, $\beta_0 = 0$, $\sigma_0 = 0$. Here and in Fig. 3, thin vertical and horizontal lines show the position at which the power of radiation attains the threshold value $P_{cr} \approx 3.9$.

Figure 2 presents the dynamics of the amplitude $|\Psi(z,n)|$ of the wave beam in the one-dimensional array of equidistantly arranged coupled active optical waveguides with the gain $\gamma = 10^{-3}$ for different initial powers of the wave beam P_0. The dashed curve shows the dependence of the radiation power P on the evolution variable z, the solid curve shows the evolution of the root-mean-square beam size $\langle a\rangle$, obtained by solving the system of equations (3), the dotted curve presents the dynamics of the beam width, calculated basing on the system of equations (11). Thin vertical and horizontal lines indicate the position, at which the radiation power achieves the threshold value for one-dimensional collapse of the wave beam initially having a plane phase front ($P = P_{cr} \approx 3.9$). As noted above, in the case of a spatial soliton at the input of the medium, the size of the wave structure will adiabatically decrease until the power exceeds the threshold value $[P(z) \gtrsim P_{cr}]$. After that, the radiation becomes localised in the central optical waveguide because of the discrete collapse [12, 17]. This case is presented in Fig. 2d. From this Figure, it is seen that the beam is captured in one channel, when the power attains P_{cr} (at $z \approx 1350$).

In the case when the initial power P_0 of the radiation is smaller than the soliton power P_{sol} (Figs 2a–2c), the diffraction is a dominating process at the initial stage of the beam evolution. The increase in the transverse size of the wave field will continue until the nonlinearity and diffraction compensate for each other, which provides the formation of a spatial soliton, i.e., until the condition $4/a^3 \approx 4P/(\sqrt{8\pi}\,a^2)$ becomes valid. Then the transverse size of the beam will exponentially decrease until the radiation power exceeds the critical one for self-focusing. The radiation becomes localised in the central optical waveguide because of the discrete collapse development. For the initial powers above the soliton power (Figs 2e and 2f) the nonlinear term is dominant, which leads to a decrease in the length L_h of the radiation capture in a single optical waveguide. In this case, in the dynamics of the amplitude distribution width the beats are observed.

Note that in the process of self-capturing of radiation in the central optical waveguide the radiation losses are considerably reduced, in contrast to the radiation collapse in the conservative case, when the leakage of part of radiation from the main region of field localisation is more essential [12]. Apparently, this is due to the adiabaticity of the beam size decrease and smoothness of the radiation entrance into one optical waveguide. Therefore, the introduction of weak gain allows efficient capture of all radiation in one optical waveguide.

4. Controlling the position of the wave beam in the lattice of optical waveguides

Now let us proceed to the analysis of self-action of a wave beam injected at an angle ($\sigma_0 \neq 0$) to the axis of the one-dimensional lattice of weakly coupled active optical waveguides, the initial power being smaller than the threshold value ($P_0 < P_{cr}$). As shown in Ref. [17], the presence of a nonzero transverse wavenumber ($\sigma_0 \neq 0$) in the conservative discrete problem leads to a different effect of the intensity maximum trajectory deviation from a straight line for the wave beams with the power exceeding the critical self-focusing value. Obviously, in a sufficiently long active medium one should expect the manifestation of such an effect even for the wave beams with the initial power smaller than the critical value ($P_0 < P_{cr}$).

First, let us consider the results of the numerical simulation. Figure 3 presents the dynamics of the amplitude enve-

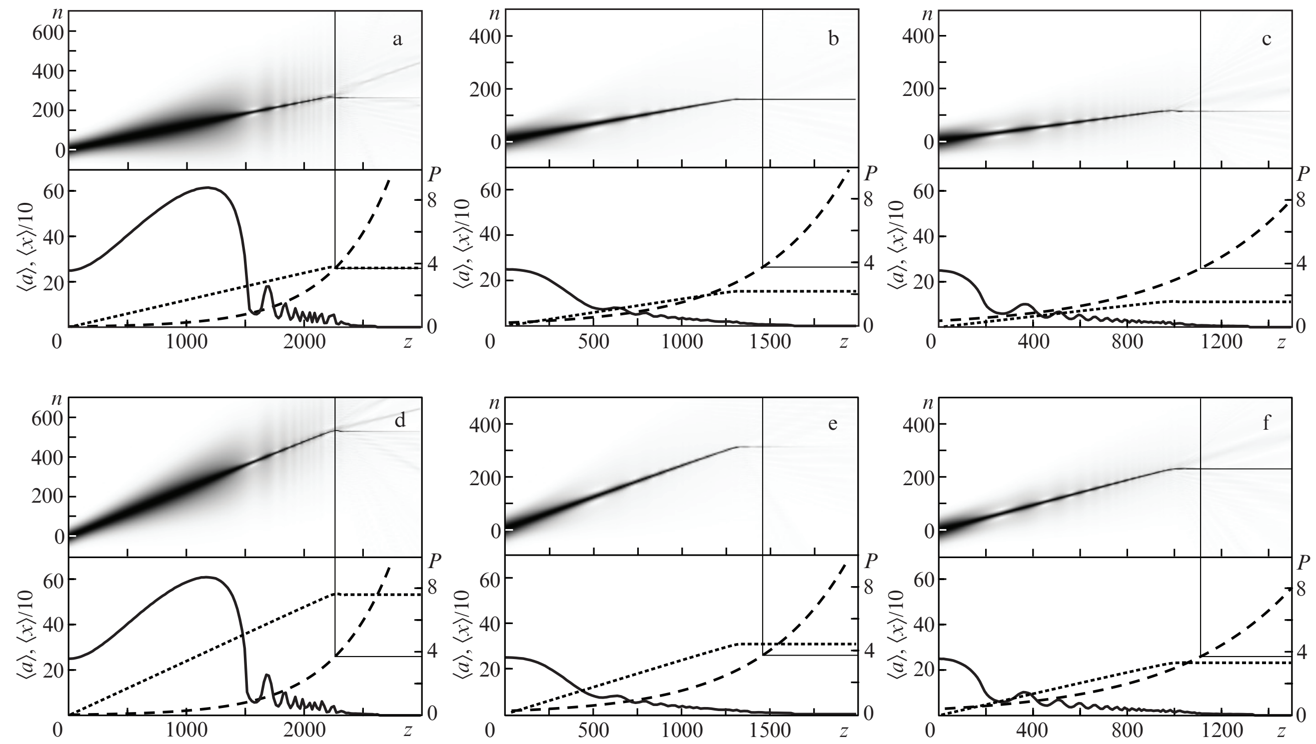

Figure 3. Dynamics of the wave beam amplitude envelope $|\Psi(z,n)|$ and z-dependences of the power P (dashed lines), beam size $\langle a\rangle$ (solid lines), and the coordinate of the beam intensity centre $\langle x\rangle$ (dotted lines) on z at (a) $P_0 = 0.2P_{sol}$, $\sigma_0 = 0.06$; (b) $P_0 = P_{sol}$, $\sigma_0 = 0.06$; (c) $P_0 = 2P_{sol}$, $\sigma_0 = 0.06$; (d) $P_0 = 0.2P_{sol}$, $\sigma_0 = 0.12$; (e) $P_0 = P_{sol}$, $\sigma_0 = 0.12$; and (f) $P_0 = 2P_{sol}$, $\sigma_0 = 0.12$; $\gamma = 10^{-3}$, $a_0 = 25$, $\beta_0 = 0$.

lope of the wave beam $|\Psi(z,n)|$ in the one-dimensional array of weakly coupled active optical waveguides with the gain $\gamma = 10^{-3}$ at different initial powers P_0 and transverse wavenumbers σ_0. At the input of the nonlinear medium, the Gaussian-shaped beam (6) was injected with the initial size $a_0 = 25$ and plane phase front ($\beta_0 = 0$). The solid curve illustrates the dependence of the root-mean-square size of the wave beam (18) on the evolution variable z, and the dotted curve plots the dependence of the intensity centre of the beam on z:

$$\langle x(z)\rangle = \frac{1}{P(z)}\sum_{n=-\infty}^{+\infty} n\,|\Psi(z,n)|^2. \qquad (19)$$

The cases presented in Figs 3b and 3e correspond to the injection of the soliton-shaped wave beam into the array of optical waveguides. From the Figures it follows that, as above (see Fig. 2), the behaviour of the wave beam size $\langle a(z)\rangle$ is independent of the initial transverse wavenumber σ_0 and is determined only by the ratio P_0/P_{sol}. In particular, in the case $P_0/P_{sol} < 1$ (Figs 3a and 3c) the size of the wave beam at the initial stage increases until the diffraction and the nonlinearity compensate for each other. Then the transverse size of the wave beam will exponentially decrease in a similar way as in Figs 3b, 3c, 3e and 3f.

At the final stage of the wave field self-action, the discreteness of the studied system begins to play the key role, essentially affecting the dynamics of the main wave beam parameters [the size $\langle a(z)\rangle$ and the coordinate of the intensity centre $\langle x(z)\rangle$], when the radiation power exceeds the threshold value P_{cr} because of gain. As a result of the discrete collapse development, the wave field is localised in the region having the size comparable with the period of the lattice of equidistantly arranged optical waveguides, which essentially affects the transverse shift of the beam with respect to the considered lattice. In contrast to the continuous medium, the ‘above-critical’ beams decline from the initial direction of rectilinear propagation and localise in a structure element, which is shifted with respect to the optical waveguide, initially central for the symmetric amplitude distribution.

As follows from Fig. 3, the final shift of the intensity centre of the wave field Δx_0 depends on the ratio P_0/P_{sol}. The maximal shift is attained at $P_0/P_{sol} < 1$, which is due to an increase in the wave beam size at the initial stage. Alongside with that, the shift of the beam also increases with increasing initial transverse wavenumber σ_0.

Now let us find the final shift of the intensity centre the wave beam Δx_0 transverse to the lattice. In the case of conservative discrete problem, this shift is inversely proportional to the square root of the radiation power ($\Delta x_0 \propto 1/\sqrt{P_0}$) [17]. Unfortunately, in the considered spatially inhomogeneous medium with gain it appears impossible to find the exact value of the shift because of the absence of the integral of the problem. The major part of the intensity centre of the shift of the wave field falls at the region of an adiabatic decrease in the wave beam transverse size.

Note that the initial transverse wavenumber σ_0 enters the analysed system of equations (11) only as a parameter, which does not change in further evolution. From the results of numerical simulation of the initial system of equations (3) it is seen that the shift Δx_0 that takes place is finite although significant as compared to the beam size (Fig. 3). For example, the total shift of the intensity centre of the wave field for a soliton-shaped beam with $\sigma_0 = 0.06$ (Fig. 3b) is smaller than

$\Delta x_0 \approx 150$, although exceeds the initial beam size by a few times. Therefore, below we restrict ourselves to the case $\sigma_0 \ll 1$ and assume that $\cos\sigma_0 \approx 1$. Therefore, the results obtained above for the self-channelling of the wave beam remain valid for the oblique injection of radiation into the active medium.

To estimate the transverse shift of the intensity centre of the soliton-shaped packet we will do the following. First, we consider the path segment, at which the power of radiation in smaller than the threshold value ($P < P_{cr}$). In this case, the discreteness of the medium does not manifest itself ($d \approx 1$) and in correspondence with Eqn (11d) the intensity centre moves along a straight line:

$$x_0 = 2\sigma_0 z. \tag{20}$$

The beam width will decrease according to law (16) until the threshold power is attained. As a result, the maximal transverse deviation of the intensity centre of the soliton-shaped wave beam from the central optical waveguide ($n = 0$) in the process of adiabatic decrease of the transverse size of the wave beam is expressed as

$$\Delta x_0 = \frac{\sigma_0}{\gamma} \ln \frac{P_{cr}}{P_0}. \tag{21}$$

To derive Eqn (21) we determined the segment of rectilinear trajectory with the length z_0, basing on the law of power increase up to the threshold $P_{cr} = P_0 \exp(2\gamma z_0)$.

At the second stage, when the radiation power exceeds the threshold, the influence of the medium discreteness becomes a determining factor. The wave field in the process of a discrete collapse is localised in the cell at the trace for $z \ll z_0$. It is easy to show that the shift of the intensity centre in this case will be also negligibly small.

Now let us compare the estimate of the final shift of the intensity centre of the wave field Δx_0 (21) with the results of numerical simulation based on the initial system of equations (3). In Fig. 4 the circles show the dependence of the final shift of the wave field in the transverse direction on the initial transverse wavenumber σ_0, obtained as a result of processing the results of numerical simulation. This case corresponds to the injection of soliton-shaped wave field into the one-dimensional array of weakly coupled active optical waveguides. The dashed line shows the estimate (21). Good agreement between the results of numerical simulation and qualitative analysis is seen.

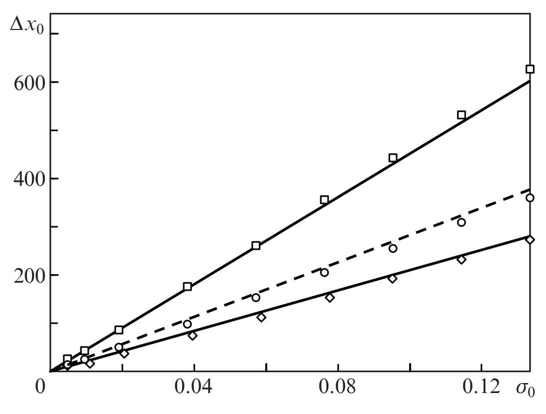

Figure 4. Final shift Δx_0 of the intensity centre of the wave beam across the considered lattice as a function of the initial transverse wavenumber σ_0 for $P_0/P_{sol} =$ (◇) 2, (○) 1 and (□) 0.2; $\gamma = 10^{-3}$, $a_0 = 25$, $\beta_0 = 0$. The curves present the results of calculation using Eqn (21).

Figure 4 presents also the dependences for the cases, when at the input a non-soliton-shaped distribution is injected (for $P_0/P_{sol} = 2$ and 0.2). One can see that in these cases the agreement between the results of numerical simulation and analytical estimate (21) is also good.

5. Conclusions

We reported a detailed analytical and numerical study of the self-action of a wave beam, injected into a one-dimensional lattice of equidistantly arranged identical weakly coupled active optical waveguides. For qualitative understanding of the basic physical processes in the considered system a variational approximation is developed that allows a closed system of ordinary differential equations (11) to be derived for the characteristic parameters of the wave field distribution (the power, the width, the phase front curvature, and the coordinate of the beam intensity maximum). This made it possible to classify the basic regimes of the wave field self-action and to determine the appropriate critical parameters.

Within the frameworks of the present approach, it was shown that the shift of the intensity centre of the wave field Δx_0 does not affect its structure changes. This fact allows separate investigation of the dynamics of the wave field injected along the axis of the considered system (when $\Delta x_0 = 0$) and the shift of the wave beam transverse to the optical waveguides. The analysis for the case of the gain increment much smaller than the inverse diffraction length of the wave field revealed the presence of two-stage dynamics. At the initial stage, the size of a soliton-shaped wave beam adiabatically decreases until the power of radiation attains the critical power for self-focusing in the one-dimensional discrete system. As the power exceeds the critical value, due to the development of discrete collapse, the wave beam becomes captured in one channel. The wave beams having the power smaller that the soliton power (14) at the initial stage will expand until the nonlinearity and diffraction compensate each other and the spatial soliton is formed. Further dynamics of the wave field is analogous to the one described above. From the analysis of Eqns (11), one can see that the adiabatic decrease in the wave beam width occurs when the phase front is nearly plane, which provides the smoothness of radiation entering into a single optical waveguide and, thus, significantly reduces the leakage of radiation from the field localisation region as compared to the case of conservative medium. The length at which the radiation is captured into the central waveguide is estimated.

The analysis of wave beam injection at an angle to the considered system has shown the following. For the wave beams, wide in comparison with the cell size, at the initial stage of propagation the variations of the wave beam structure described above are initially accompanied by quasi-rectilinear shift of the intensity centre transverse to the lattice of optical waveguides. The subsequent process of radiation capture by a single waveguide stops further shift. The estimates of the total shift are in good agreement with the results of numerical simulation. The value of the shift is mainly determined by the ratio of the injection angle to the gain increment.

Acknowledgements. The work was supported by the Russian Science Foundation (Grant No. 16-12-10472).

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