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Methods for analysing the quality of the element base of quantum information technologies

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Abstract. **Methods for analysing the quality of the element base of quantum information technologies are considered. Methods for simulating quantum operations with allowance for quantum noise, based on the formalism of quantum operations (operator sum method and Choi–Jamiolkowski isomorphism), are described. Results of simulating some one- and two-qubit quantum transformations with allowance for the amplitude and phase relaxation and the presence of depolarising noise are reported. The quantum state tomography, based on the root approach and maximum likelihood method, is considered. The method of guaranteed estimation of quantum tomography accuracy, which implies quantitative estimation of measurement results, is described. Tomography of a single-qubit transformation on a superconducting quantum processor (IBM) is performed. The practical purpose of the study is to provide high quality and efficiency of quantum information technologies. The results obtained can be used to improve the quality of logic gates of quantum computers and debug the procedures of quantum state control in quantum cryptography problems.**

Keywords: *quantum informatics, qubit, algorithm, quantum tomography, measurements, decoherentisation, simulation.*

1. Introduction

Quantum information technologies form a rapidly developing line of research. It is known that quantum calculations may reduce significantly the solution time for a number of practically important physical and mathematical problems [1–3]. Many large IT companies have been involved in the development of multiqubit quantum processors [4–6]. However, the operation accuracy of the existing prototypes of quantum calculators remains low in view of the presence of quantum noises of different nature and the absence of a proper methodology for controlling quantum states and processes. To solve this problem, it is necessary, on the one hand, to improve technological equipment and noise suppression methods and, on the other hand, to elaborate efficient methods for monitoring quantum processes and quantum states.

Simulation of ideal quantum systems is performed in the language of pure vector states and unitary transformations. Due to the interaction of a system under study with its environment, the description must be performed in the language of density matrices [3]. In this case, the state evolution is not unitary and can be described using either a solution to the Lindblad equation [7] or the quantum operations formalism, which is more general [8]. In this study, we present an approach to the analysis of noisy quantum transformations that is based on the Choi–Jamiolkowski isomorphism [9–11]. This approach makes it possible to analyse characteristic quantum gates as a whole, independently of the states they act on.

Along with mathematical simulation of quantum transformations, it is necessary to monitor them under real experimental conditions. The main monitoring tool is tomography of quantum states and processes [3, 12–19], which serves to link the development of the element base of quantum computers with its practical implementation. Methods of numerical analysis and statistical simulation with allowance for the quantum noise influence and the results of technological and experimental studies in this field make it possible to estimate exhaustively the quality and efficiency of designed quantum registers and formulate requirements to the experimental equipment and technology. Due to the feedback, the advanced approach allows one to deal in the best way with the existing resources in order to optimise the development of quantum information technologies.

This work reflects the experience gained at the Valiev Institute Physics and Technology of the Russian Academy of Sciences in developing methods that ensure high quality and efficiency of quantum information technologies. The research in this field is performed in close cooperation with the physicists from M.V. Lomonosov Moscow State University.

2. Simulation of noisy quantum transformations

An ideal quantum gate is known to provide a unitary transformation of a quantum state. However, the evolution of a

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real state is never unitary. In more realistic models it is necessary to take into account the inevitable interaction between a quantum system and its environment. Within the theory of open quantum systems with allowance for the Markovian approximation, where the environment is characterised by a sufficiently short 'memory time' in comparison with the quantum operation time, the evolution of a state (determined by a density matrix $\rho_{\rm in}$) is set by the operator formula [3, 8, 9]

$$\rho_{\rm out} = \mathcal{E}(\rho_{\rm in}) = \sum_k E_k \rho_{\rm in} E_k^{\dagger}, \tag{1}$$

where E_k are the so-called Kraus operators, which satisfy the normalisation condition (trace preservation)

$$\sum_k E_k^{\dagger} E_k = I .$$

It can be shown that an arbitrary quantum operation is described by a density matrix in a space of larger dimension (in comparison with the space of matrices $\rho_{\rm in}$). This isomorphic representation of operations via density matrices is referred to as the Choi–Jamiolkowski isomorphism. To construct the Choi–Jamiolkowski correspondence state for a quantum operation acting in an s-dimensional Hilbert space ($s = 2^n$, where n is the number of qubits in the system), one must take the maximally entangled state

$$\frac{1}{\sqrt{s}} \sum_{i=1}^{s} |i\rangle \otimes |i\rangle \tag{2}$$

and apply the quantum operation under consideration to the second subsystem:

$$\rho_{\chi} = \frac{1}{s} \sum_{i_1, i_2 = 1}^{s} |i_1\rangle\langle i_2| \otimes \mathcal{E}(|i_1\rangle\langle i_2|) . \tag{3}$$

The Choi–Jamiolkowski correspondence state contains complete information about the corresponding quantum transformation. Due to the above-considered isomorphism, quantum schemes can be analysed without binding them to the input state. The measure of operation accuracy is taken to be the probability of coincidence of the correspondence states of noisy and ideal schemes. Computationally, the simulation of Choi–Jamiolkowski states is more difficult than the simulation of the evolution of individual quantum states, because they are set in a space of dimension s^2; however, this simulation makes it possible to exclude from consideration various input states of quantum operation, along with subsequent solution of the optimisation problem.

The correspondence matrix can be set in different representations, which are determined by different sets of basis matrices. It is often convenient to present it in the basis of Pauli matrices (specifically this representation will be used below), because this basis, being the most fundamental one, provides a very descriptive visualisation of the correspondence matrix.

The principles of simulating noisy quantum gates and schemes were described in detail in our previous studies [11, 19].

From the practical point of view, it is important to study the influence of the processes of amplitude and phase relaxation (characterised by the parameters T_1 and T_2, respectively) on the system under consideration, because they naturally occur in many physical realisations of quantum calculations. Figure 1 shows the results of simulating the Hadamard transformation in the ideal case and with allowance for the aforementioned processes. The so-called chi matrix χ is shown, which coincides (accurate to normalisation) with the Choi–Jamiolkowski state.

A similar analysis can be performed for more complex systems. Figure 2 demonstrates the influence of amplitude–phase relaxation on a two-qubit controlled NOT (CNOT) transformation. In addition, the influence of the so-called depolarising noise, which is also characteristic of many physical systems, is considered.

3. Quantum state tomography

According to Bohr's complementarity principle, to obtain complete information about a quantum state, it is necessary to perform a series of mutually supplementing quantum measurements on an ensemble of identically prepared representatives of this state. The proposed approach is based on the concept of adequacy, completeness, and accuracy of quantum measurements, which was considered in detail in [17–21].

A set of m projection measurements of a pure state, described by a column vector of length s, can be presented in the matrix form:

$$M = Xc. \tag{4}$$

Here, M is a column vector of length m, which describes the probability amplitudes of obtaining different measurement results, and X is the so-called instrumental matrix of size $m \times s$; each row of this matrix is a bra state vector, onto which projecting is performed. The probabilities (per unit time) of obtaining the corresponding measurement results are set by squared moduli of amplitudes:

$$\lambda_i = |M_i|^2,\ i = 1, 2, \ldots, m. \tag{5}$$

If each jth protocol row is measured independently of other rows during the exposure time t_i, the measurement results is described by the Poisson statistics with a mean $\lambda_j t_j$.

After accumulating the measurement statistics, the quantum state is reconstructed using the root approach and maximum likelihood method [15, 22, 23], which yield the following quasi-linear equation for the column vector c (likelihood equation):

$$\tilde{I}c = J(c)c,\ \tilde{I} = \sum_{j=1}^{m} t_i X_j^{\dagger} X_j,\ J(c) = \sum_{j=1}^{m} \frac{k_j}{\lambda_j(c)} X_j^{\dagger} X_j, \tag{6}$$

where X_j is the jth row of instrumental matrix X and k_j is the number of events registered for this row.

When carrying out tomography of unknown quantum states, there arises a problem of choosing the optimal quantum measurement protocol. The most widespread protocols include those implying expansion of unity:

$$\sum_{j=1}^{m} t_j X_j^{\dagger} X_j = {\rm const} \cdot I_s,$$

where I_s is a unit matrix of size $s \times s$. In the case of single-qubit states, it is natural to choose protocols based on the symmetry of regular polyhedra: the ends of the protocol Bloch vectors

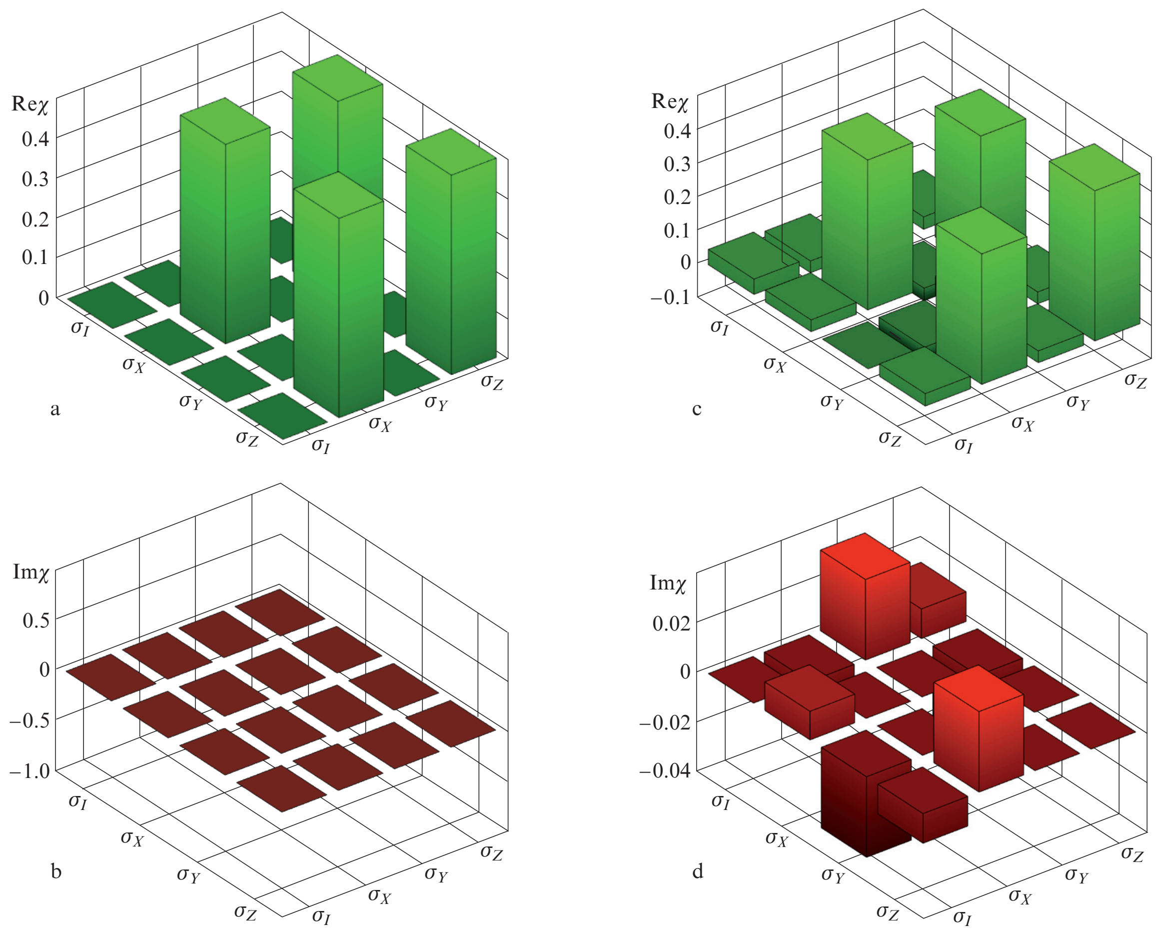

Figure 1. (a, c) Real and (b, d) imaginary parts of the chi-matrix for an Hadamard gate: (a, b) ideal case and (c, d) the effect of amplitude–phase relaxation with the parameters $T_1 = 3$ and $T_2 = 5$ (the operation time is chosen to be unity); σ_I is a unit matrix of size 2×2.

lie at the centres of faces of correct polyhedra circumscribed around the Bloch sphere [18]. The simplest protocol consisting of four elements is that based on the tetrahedron symmetry [16, 18] (Fig. 3a). Here, different states are reconstructed with different degrees of accuracy, as demonstrated in Fig. 3b. Characteristics of other protocols were also considered in [18], and optimal protocols (obtained by solving minimisation problems) were developed in [24].

Among the quantum measurement protocols that are not reduced to expansion of unity, we should pay attention to Lorentz protocols [25]. Lorentz quantum measurements of mixed qubit states are based on the consideration of four-dimensional pseudorotations, similar to the transformations in the special theory of relativity. These nonunitary transformations of qubits are a generalisation of the well-known unitary rotations on the Bloch sphere. Note that Lorentz measurements also allow for the generalisation to the case of multiqubit states. It is important that Lorentz protocols of quantum measurements may provide a higher accuracy of tomography than any conventional protocols that are reduced to expansion of unity.

In the case of tomography of N-qubit states, each qubit is measured separately using the instrumental matrix X, and the general instrumental matrix for a quantum state is set by the corresponding tensor product $X^{\otimes N}$. The tomography accuracy for arbitrary specified protocols of measurements and quantum states can be theoretically estimated based on the quantum generalisation of the Fisher information matrix. In the case of tomography of pure states that is considered here, this approach yields the following (real, symmetric, and positively defined) complete information matrix of $2s \times 2s$ size [17]:

$$H = 2\sum_{j=1}^{m} \frac{t_j}{\lambda_j} (\tilde{X}_j^{\dagger} \tilde{M}_j)(\tilde{X}_j^{\dagger} \tilde{M}_j)^{\dagger}, \ \tilde{M}_j = \begin{pmatrix} \mathrm{Re}\, M_j \\ \mathrm{Im}\, M_j \end{pmatrix},$$

$$\tilde{X}_j = \begin{pmatrix} \mathrm{Re}\, X_j & -\mathrm{Im}\, X_j \\ \mathrm{Im}\, X_j & \mathrm{Re}\, X_j \end{pmatrix}. \quad (7)$$

An analysis of the information matrix makes it possible to estimate quantitatively the fluctuations of quantum state parameters and, therefore, the accuracy of reconstructing an unknown state [17, 18]. Figure 4 shows the results of computed tomography experiments on a two-qubit quantum state and the theoretical distribution obtained based on matrix (7). The chi-square criterion shows good correspondence between the theory and numerical experiment.

The above consideration is valid not only for pure states but also for mixed states of arbitrary rank. The corresponding consideration can be performed using the purification procedure (where a mixed state is complemented to pure in a space of higher dimension) [17].

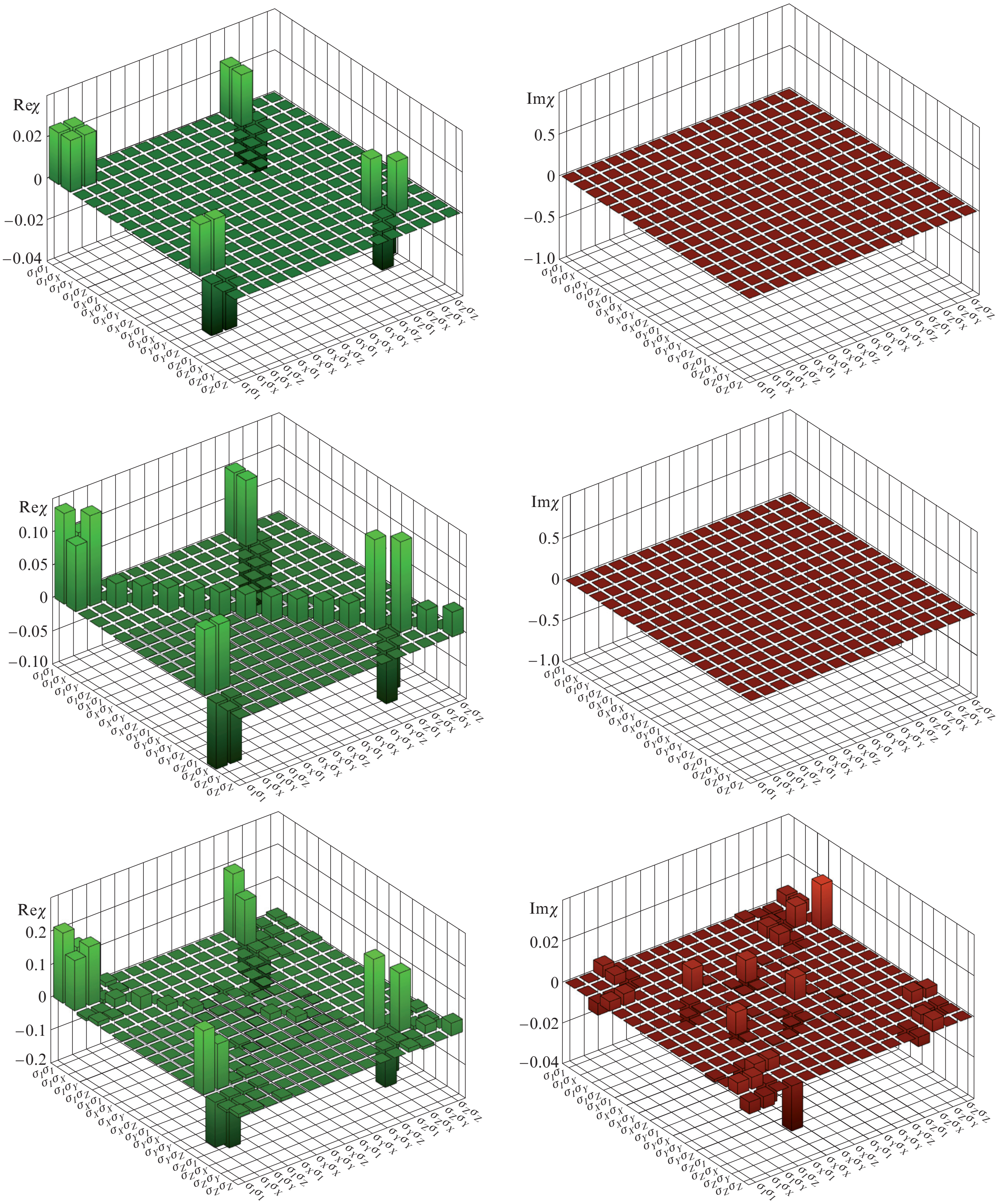

Figure 2. Real (left column) and imaginary (right column) parts of the chi-matrix for a CNOT gate: (upper row) ideal case, (middle row) the effect of depolarising noise with the noisiness level $p = 0.6$, and (lower row) the effect of amplitude–phase relaxation with the parameters $T_1 = 5$ and $T_2 = 3$ (the operation time is chosen to be unity).

4. Quantum process tomography

The above-described methods can be used to determine a transformation carried out by some unknown quantum channel. This procedure is referred to as quantum process tomography [3, 26, 27]. It plays the most important role in providing the quality of quantum information technologies, because it allows one to characterise completely the quantum gates

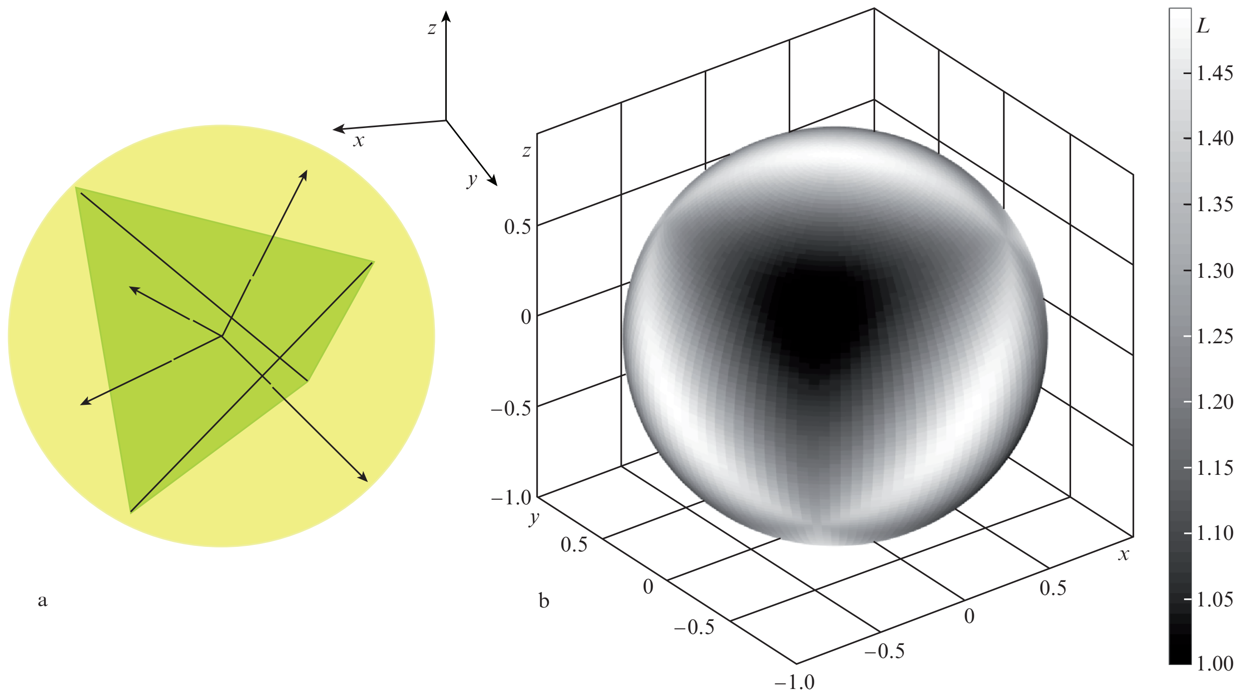

Figure 3. (a) Protocol based on the tetrahedron symmetry and (b) the theoretical distribution of the loss function $L = l\langle 1 - F\rangle$ (l is the sample volume) for a pure quantum state on the Bloch sphere (tetrahedron protocol), obtained based on the analysis of information matrix (7).

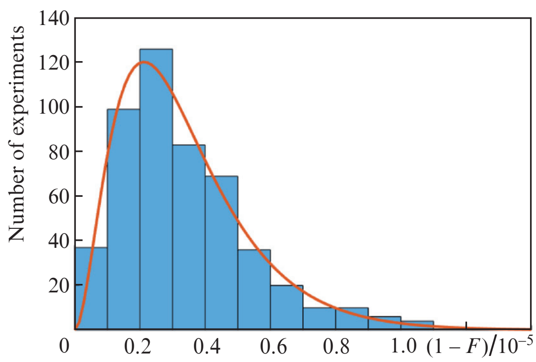

Figure 4. Results of computed tomography experiments on a two-qubit entangled state $|00\rangle + i|11\rangle)/\sqrt{2}$, using a tetrahedron protocol (columns). The solid curve shows the theoretical distribution of tomography accuracy loss, constructed with application of information matrix (7). In total, 500 experiments were performed, with a sample volume $l = 10^6$ in each experiment.

implemented in a real physical system, detect errors in them, and then introduce (based on the obtained results) corresponding corrections to the technology and tuning of the control equipment.

This approach was used to analyse the behaviour of a five-qubit superconducting quantum processor ibmqx4 (IBM) [4], which is free online. In particular, we performed tomography of a single-qubit phase-shift gate Z on the processor qubit Q1. The tomographic protocol was chosen to be the cube protocol, because its realisation on the processor turned out to be most convenient. To carry out tomography of a quantum process corresponding to the gate Z, six basis states were applied to the input:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},\ |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},\ |+\mathrm{i}\rangle = \frac{|0\rangle + \mathrm{i}\,|1\rangle}{\sqrt{2}},$$

$$|-\mathrm{i}\rangle = \frac{|0\rangle + \mathrm{i}\,|1\rangle}{\sqrt{2}},\ |0\rangle,\ |1\rangle.$$

The initial zero state was transformed into the aforementioned basis states by adding the necessary rotating gate to the scheme input. At the output, qubit measurements were performed in three bases, corresponding to the σ_X, σ_Y, and σ_Z observables (Pauli matrices). In total, it was necessary to apply 18 different quantum schemes to perform single-qubit gate tomography. Each scheme was initialised 8192 times to accumulate measurement statistics. The two-qubit quantum Choi–Jamiolkowski state for gate Z was reconstructed based on the measurement data.

An adequate model was developed for the quantum process [17, 23]. It was shown that the model corresponding to the transformation rank $r = 2$ is statistically significant at a significance level of 5%. The fidelity F between the second-rank chi-matrix (Fig. 5) and the chi-matrix of ideal transformation Z turned out to be ~93.29%. Figure 6 shows the fiducial distribution of tomography accuracy for the case under consideration, calculated using the complete information matrix (7). This distribution, providing guaranteed accuracy of statistical estimation of quantum operation, answers the question about the proximity of reconstructed quantum operation to the unknown true quantum operation realised on the ibmqx4 processor. It can be seen in Fig. 6 that the goodness of fit is indeed very high. With a confidence probability of 95% (the unhatched region in Fig. 6), one can state that the fidelity F for an unknown accurate quantum operation is no less than 99.84%. The proximity of this value to 100% indicates that the significant difference between the reconstructed chi-

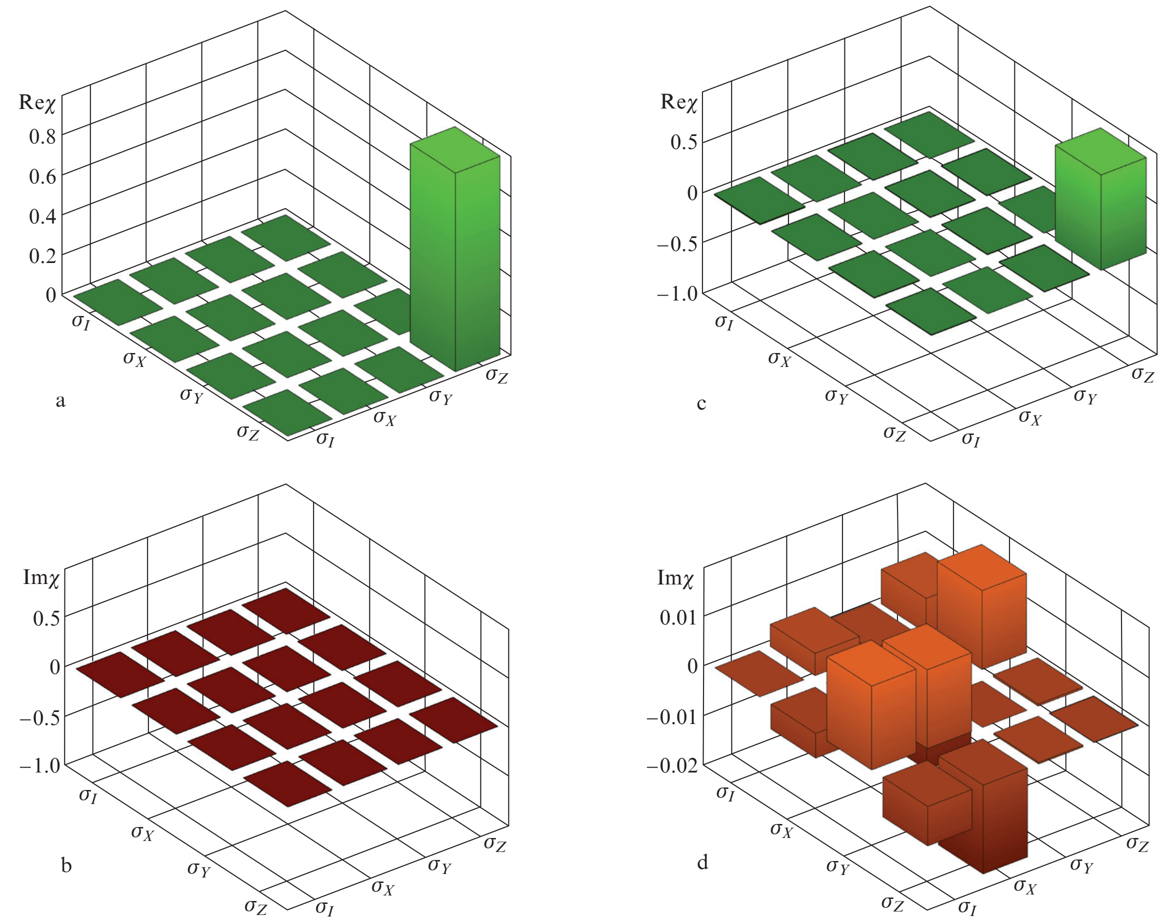

Figure 5. (a, c) Real and (b, d) imaginary parts of the chi-matrix for a gate Z: (a, b) ideal case and (c, d) the chi-matrix obtained by tomography of a gate based on an ibmqx4 quantum processor (qubit Q1, IBM). A cube protocol was used; the total sample volume is 18432 representatives of the quantum statistical ensemble.

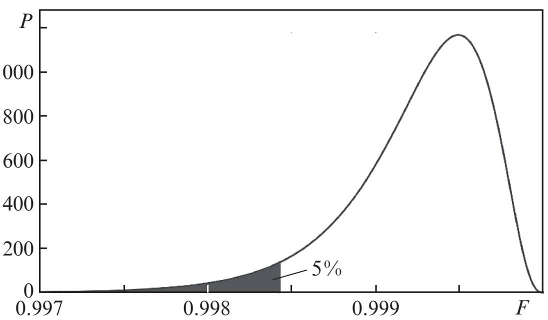

Figure 6. Fiducial distribution P of fidelity F between the developed model for a gate Z of ϕт ibmqx4 quantum processor and an unknown accurate quantum operation.

matrix and the ideal matrix (Fig. 5) is almost completely due to the real decoherentisation processes occurring in the IBM quantum processor rather than to the statistical reconstruction error.

Note that tomography of different transformations performed by an IBM processor was also considered in [28]. However, in our opinion, the procedures developed in that study are not quite correct and have an obviously low accuracy. For example, the trace-preservation condition $\mathrm{Tr}_1(\chi) = I_s$ (Tr_1 is the partial trace over the first subsystem), which is imposed on any chi-matrix, was disregarded in [28]. In addition, reconstruction was carried out using the pseudoinversion method, which does not always yield physically meaningful results and obviously cannot provide asymptotic efficiency and guaranteed accuracy of statistical estimates [23].

5. Conclusions

The methodology of simulating quantum operations in order to develop the element base of quantum computers was considered. Different ways to describe quantum operations, including the operator sum method and representation via the Choi–Jamiolkowski state and the corresponding chi-matrix, were presented. The processes of amplitude and phase relaxation of qubit states and their simultaneous Hamiltonian dynamics were taken into account.

The quantum state and process tomography based on the root approach and maximum likelihood method is described. This approach guarantees an accuracy close to the fundamental physical statistical limit. The use of mathematical statistics methods allows one to perform a complete analysis of information obtained in these measurements and determine the attainable level of tomography accuracy.

A method of guaranteed estimation of accuracy in realisation of quantum operations, based on quantitative estimation of measurement data, was developed. This method was applied to estimate the quality of transformations performed by a superconducting quantum processor manufactured by IBM.

The proposed method makes it possible to estimate comprehensively the quality and efficiency of designed quantum registers and formulate requirements to the experimental equipment and technology. Due to the feedback, the developed approach allows one to deal in the best way with the existing resources in order to optimise the development of quantum information technologies. The efficiency of this approach has been demonstrated by us in the last years when working with optical and superconducting qubits [17–19, 27].

References

1. Valiev K.A., Kokin A.A. *Kvantovye komp'yutery: nadezhda i real'nost'* (Quantum Computers: Expectations and Reality) (Izhevsk: RKhD, 2001).
2. Bogdanov Yu.I., Valiev K.A., Kokin A.A. *Mikroelektron.*, **40**, 243 (2011).
3. Nielsen M.A., Chang I.L. *Quantum Computation and Quantum Information* (Cambridge: Cambridge Univ. Press, 2010).
4. https://www-03.ibm.com/press/us/en/pressrelease/51740.wss.
5. https://newsroom.intel.com/news/intel-delivers-17-qubit-superconducting-chip-advanced-packaging-qutech/.
6. https://research.googleblog.com/2018/03/a-preview-of-bristlecone googles-new.html.
7. Lindblad G. *Commun. Math. Phys.*, **48** (2), 119 (1976).
8. Kraus K. *Lect. Notes Phys.*, **190**, 154 (1983).
9. Holevo A.S. *Quantum Systems, Channels, Information. A Mathematical Introduction* (Berlin: De Gruyter, 2013).
10. Preskill J. *Kvantovaya informatsiya i kvantovye vychisleniya* (Quantum Information and Quantum Computation) (Moscow, Izhevsk: NITs 'Regulyarnaya I Khaoticheskaya Dinamika', Institut Komp'yuternykh Issledovanii, 2008) Vol. 1.
11. Bogdanov Yu.I., Chernyavskii A.Yu., Holevo A.S., Lukichev V.F., Nuyanzin S.A., Orlikovskii A.A. *Trudy FTIAN*, **22**, 39 (2012).
12. D'ariano M., Paris M.G.A., Sacchi M.F. *Adv. Imag. Elect. Phys.*, **128**, 205 (2003).
13. Lvovsky A.I., Raymer M.G. *Rev. Mod. Phys.*, **81** (1), 299 (2009).
14. Bogdanov Yu.I. *Opt. Spektrosk.*, **96** (5), 735 (2004).
15. Banaszek K., D'Ariano M., Paris M.G.A., Sacchi M.F. *Phys. Rev. A*, **61**, 010304 (2000).
16. Rehacek J., Englert B.-G., Kaszlikowski D. *Phys. Rev. A*, **70**, 052321 (2004).
17. Bogdanov Yu.I. *JETP*, **108** (6), 928 (2009) [*Zh. Eksp. Teor. Fiz.*, **135** (6), 1068 (2009)].
18. Bogdanov Yu.I., Brida G., Bukeev I.D., Genovese M., Kravtsov K.S., Kulik S.P., Moreva E.V., Soloviev A.A., Shurupov A.P. *Phys. Rev. A*, **84**, 042108 (2011).
19. Bogdanov Yu.I., Kalinkin A.A., Kulik S.P., Moreva E.V. *New J. Phys.*, **15**, 035012 (2013).
20. Bogdanov Yu.I., Kulik S.P. *Laser Phys. Lett.*, **10** (12), 125202 (2013).
21. Bogdanov Yu.I., Avosopyants G.V., Belinskii L.V., Katamadze K.G., Kulik S.P., Lukichev V.F. *JETP*, **150** (2), 212 (2016)] [*Zh. Eksp. Teor. Fiz.*, **150** (2), 246 (2016)].
22. Fisher R.A. *Philos. Trans. R. Soc. London. Ser. A*, **222**, 594 (1922).
23. Borovkov A.A. *Matematicheskaya statistika. Otsenka parametrov and proverka gipotez* (Mathematical Statistics: Estimation of Parameters and Testing Hypotheses) (Moscow: Nauka, 1984).
24. Bogdanov Yu.I., Belinskii L.V. *Trudy FTIAN*, **25**, 90 (2015).
25. Bogdanov Yu.I., Bogdanova N.A. *Proc. Int. Conf. on Laser Physics 2017* (Kazan, Russia, 2017).
26. Chuang I.L., Nielsen M.A. *J. Mod. Opt.*, **44** (11-12), 2455 (1997).
27. Bogdanov Yu.I., Bantysh B.I., Kalinkin A.A., Kulik S.P., Moreva E.V., Shershulin V.A. *JETP*, **145** (6), 963 (2014) [*Zh. Eksp. Teor. Fiz.*, **145** (6), 963 (2014)].
28. Shukla A., Sisodia M., Pathak A. *Complete Characterization of the Single-qubit Quantum Gates Used in the IBM Quantum Processors*; arXiv:1805.07185 (2018).