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On mathematical and physical approaches to constructing a quantum cluster state in continuous variables, or is it possible to construct a cluster from different modes?

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Abstract. **We consider the problems of constructing a cluster state from a set of orthogonal modes. It is shown that the commonly used unitary transformation is not reduced in this case to a set of standard operations of linear optics performed over fields in a squeezed state, but should be supplemented by a more complicated operation of a noise-free transformation of the quantum signal profile. At the same time, if it is possible to form a cluster whose node amplitudes have different time profiles, this property does not interfere with the performance of calculations on the cluster.**

Keywords: *quantum cluster state, orthogonal modes, squeezed state of the light.*

1. Introduction

Interest in quantum computations and, in particular, in one-way quantum computing [1–3] has generated a huge number of variants for constructing quantum cluster states. The principles of generation of quantum states and operations with them differ dramatically depending on the 'language' used to describe them, i.e. in terms of discrete [1] or continuous variables [4–6]. For both variants of the description, an apparatus has been developed for performing logical operations, which makes it possible to implement a universal quantum computer provided that a suitable multiparticle-entangled quantum state is constructed. In this paper, we will consider only continuous-variable cluster states.

A quantum 'resource' for constructing a quantum cluster state can be quadrature squeezing of light modes [7, 8], in which case one can speak of temporal mode squeezing or of spatial squeezing. Methods for constructing clusters based on correlations over the orbital angular momentum have been proposed in Refs [9, 10]. There are proposals for the construction of cluster states based on the spin waves of atomic ensembles [11, 12] and optomechanical systems [13]. Of interest are the methods for generating 'hybrid' cluster states on the basis of material and field oscillators [11].

When discussing the problem of constructing a quantum cluster state, and then calculating it, the key concept is the idea of light modes or of the modes of other considered quantum objects. Without loss of generality, we shall speak of light modes. Each mode is associated with a quantum oscillator, and the transformation of such a mode must be described by the laws of quantum mechanics. In this paper, we would like to draw attention to the issue of the correctness of the use of certain mathematical constructions in the discussion of the cluster state generation. The motivation for us was the publication of a number of papers [14–17], which consider the formation of a cluster based on orthogonal modes of different nature.

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2. Cluster generation as a unitary transformation over squeezed modes

We recall briefly how a continuous-variable cluster state is determined. The mathematical description of the procedure for constructing a cluster begins with specifying the graph of the future cluster, as it determines the topology (structure) of the state that we want to obtain at the output. To determine the graph G of the cluster state, it is sufficient to specify a set of nodes and a set of edges connecting these nodes. The nodes of the graph are the modes of a physical system described by pairs of canonical variables $\{\hat{X}_i, \hat{Y}_i\}$ satisfying canonical commutation relations, and the edges are the quantum entanglement of modes. Each edge connecting the nodes i and j is associated with a real number $v_{ij} \in [-1, 1]$, called the weight of an edge. The set of these weights defines an adjacency matrix $V = \{v_{ij}\}$, which completely defines the graph G.

Usually, when constructing a cluster state, the following procedure for its generation is discussed. We consider n independent quantum harmonic oscillators in a quadrature-squeezed state. Each of these oscillators is given by quadrature operators $\hat{x}_i$ and $\hat{y}_i$, subject to the canonical commutation relations

$$[\hat{x}_j, \hat{y}_k] = \frac{\mathrm{i}}{2}\delta_{jk}, \tag{1}$$

where the subscripts j and k denote the corresponding oscillator, and δ_{jk} is the Kronecker delta. In this case, we assume that all the oscillators are squeezed along the $\hat{y}$-quadrature [18], i.e., their variances are less than those of the vacuum state:

$$\langle \delta\hat{y}_i^2 \rangle < \frac{1}{4}, \quad i = 1,...,n.$$

Let us entangle these subsystems so that the coupling strength between the ith and jth oscillators corresponds to the element v_{ij} of the adjacency matrix V [in the sense of satisfying equalities (3) and (4), see below]. As a result, we obtain a physical system in the quantum cluster state, to which the graph G given by the adjacency matrix V corresponds. Note that the nodes of the graph G correspond to quantum harmonic oscillators entangled in a certain way. Such an entanglement can be described by the Bogoliubov transformation [19] over an original set of independent quadrature-squeezed oscillators:

$$\hat{X}_j + \mathrm{i}\hat{Y}_j = \sum_{k=1}^{n} u_{jk}(\hat{x}_k + \mathrm{i}\hat{y}_k), \quad j = 1,...,n, \tag{2}$$

where $U = \{u_{ij}\}_{j,k=1}^{n}$ is a unitary matrix fixing a specific set of transformations over subsystems, so that the result of the transformation corresponds to the adjacency matrix V (we will discuss the connection of these matrices below); and $\hat{X}_j$ and $\hat{Y}_j$ are the quadrature operators of the jth node of the cluster state.

The quantum-statistical properties of discrete-variable clusters are usually described using stabilisers [20]. However, for continuous variables, the most natural way to describe the statistics of a cluster state is a 'nullifier' operator, which is introduced for each node of the graph G:

$$\hat{N}_j = \hat{Y}_j - \sum_{i=1}^{n} v_{ji}\hat{X}_i, \quad j = 1,...,n. \tag{3}$$

By definition, the physical system is in the quantum cluster state if the variances of all its nullifiers tend to zero in the limit of infinite squeezing of the quantum harmonic oscillators used to generate it [18]:

$$\forall j = 1,...,n, \ \lim \langle \delta\hat{N}_j^2 \rangle = 0 \text{ at } \langle \delta\hat{y}_1^2 \rangle \to 0,..., \langle \delta\hat{y}_n^2 \rangle \to 0. \tag{4}$$

According to the Bloch–Messiah reduction theorem [21], an arbitrary Bogoliubov transformation can be represented in the form of four successive operations:

$$\hat{U} = \hat{R}(\phi_1)\hat{S}(r)\hat{R}(\phi_2)\hat{D}(\alpha), \tag{5}$$

where $\hat{R}(\phi_1)$ and $\hat{R}(\phi_2)$ are the rotation operators on the family of angles $\phi_1 = \{\phi_{1ij}\}_{i,j=1}^{n}$ and $\phi_2 = \{\phi_{2ij}\}_{i,j=1}^{n}$, respectively. The operator $\hat{S}(r)$ corresponds to the n-mode squeezing with real squeezing coefficients $r = \{r_1..,r_n\}$. The operator $\hat{D}(\alpha)$ is the displacement operator of n quadratures by the quantities $\alpha = \{\alpha_1,...,\alpha_n\}$. If the transformation is performed over Gaussian states (we consider this case), the above operators can be represented in matrix form: for example, rotation and squeezing matrices for a single-mode state have the form

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad S(r) = \begin{pmatrix} \mathrm{e}^{-r} & 0 \\ 0 & \mathrm{e}^{r} \end{pmatrix}. \tag{6}$$

In the case of the n-mode state, the corresponding matrices will acquire a block structure. Note that the matrix of quadrature displacements present in the general expansion can be omitted in further transformations, since the displacements of each mode described by it for some given classical quantities do not affect the quantum-statistical properties of the cluster and its use for quantum computations. As was shown in [22], when performing the procedure for constructing a cluster from initially squeezed modes, it suffices to confine oneself to linear optics transformations. The matrix U in this case corresponds to a successive application of rotation and mixing transformations on a beam splitter. The condition for the number of required operations can be obtained based on the cluster dimension and topology [23]. However, important for us is the mode mixing procedure 'sewed' in the Bogoliubov transformation. The mixing transformation cannot be performed on modes with different profiles without an addition of noise. In fact, if the field with the time profile $L_1(t)$ is incident on the first input of the beam splitter [and with the time profile $L_2(t)$ on the second], then, according to the transformation rules, we should consider the field of the same mode at the other input of the beam splitter. Since there is no backillumination of this mode at the second input of the beam splitter, this field should be considered as a vacuum field. Thus, the mixing of the fields with different profiles should in fact be described as a four-wave linear process in which two vacuum fields and two squeezed fields participate. When such fields are mixed, the mode correlations decrease, and the obtained field (although exhibits entanglement properties) is small due to the influence of vacuum noise.

In other words, speaking of the unitary multimode Bogoliubov transformation, we must take into account also the modes initially in the vacuum state and not participating in the process. The rotation of the basis causes the interaction of these modes.

Thus, we see that, by mixing modes with different time profiles, it is impossible to generate a well-entangled state. In particular, one cannot obtain a quantum cluster state by linear transformations over squeezed modes with different time profiles. The same conclusion is valid if the modes differ not by the time profiles, but by any other characteristic that allows one to speak of the mode composition of radiation.

Using the Bogoliubov transformation allows us to discuss the problem of mode mixing not directly, like the transformation of light fields on a beam splitter, but as some generalised procedure that transforms the forms of the mixed fields. As an example of such a procedure, we can mention the writing of a signal in the cell of quantum memory with the subsequent reading of a signal of a different form from this cell [24, 25]. If the efficiency of such a process tends to unity, the quantum-statistical correlations of the initial wave are transferred without losses to a wave with a different mode profile. As is well known, this transformation is described by the beam splitter Hamiltonian, and the high efficiency of the process corresponds in this analogy to the transmission coefficient of the beam splitter, tending to unity. However, it should be noted that the creation of such converters requires considerable effort, and it is unreasonable to assume the presence of a converter in the 'default' scheme, without discussing options for its implementation.

We have discussed the issue of constructing a cluster state by linear transformations over squeezed oscillators. However, this discussion does not solve the problem, since it excludes the possibility of using nonlinear transformations to construct a cluster. Suppose that there exists a nonlinear physical process that allows us to write down a formal set of equalities (3), where the operators $\hat{X}_j$ and $\hat{Y}_j$ refer to orthogonal modes [26]. In what sense should we understand this equality and can we speak of such a state as a cluster state?

To answer this question, let us turn again to the definition of the cluster state (3) and (4). This definition includes the averaging procedure, that is, it is directly related to the measurement. What measurement is assumed with such a writing? It can be confidently asserted that it is not possible to specify a homodyne profile that performs a measurement in the case when the quadratures $\hat{X}_j$ and $\hat{Y}_j$ belong to orthogonal modes.

Thus, in this case, we must again assume that there is an auxiliary device that transforms the mode profile with the preservation of its quantum statistics.

In connection with the fact that the quantum cluster state is of interest primarily as a resource for one-way computations, we will discuss the problem of constructing a cluster based on orthogonal modes from the point of view of the possibility of performing such calculations on it. Because the calculation procedure is based on measurements, one can expect that they also impose limitations on the cluster's mode composition.

3. One-way computations and their relation to the choice of cluster modes

Let us describe the procedure for one-way quantum computations on a many-particle quantum field state satisfying equality (3), where the operators $\hat{X}_j$ and $\hat{Y}_j$ are assumed to be related to orthogonal modes.

As an example, we consider a calculation on a linear four-mode cluster state. The calculation scheme is shown in Fig. 1. Amplitudes of such a cluster state can be represented in the form

$$\hat{A}_i(t) = L_j(t)(\hat{X}_j + \mathrm{i}\hat{Y}_j) + \mathrm{vac}_j, \quad j = 1,....,4. \tag{7}$$

Here $\{L_j(t)\}_{j=1}^{4}$ are arbitrary orthogonal functions belonging to a complete orthonormal set, over which the field amplitudes are expanded. We assume that the remaining modes, which are also present in the expansion, are in the vacuum state and are described by the terms vac_j. The quantities $\{\hat{X}_j, \hat{Y}_j\}_{j=1}^{4}$ are the operator expansion coefficients, which are the canonical quadratures of the modes. Since the quantum state in question is a cluster, these quadratures must satisfy three van Loock–Furusawa inequalities:

$$\langle \delta(\hat{Y}_1 - \hat{X}_2)^2 \rangle + \langle \delta(\hat{Y}_2 - \hat{X}_1 - \hat{X}_3)^2 \rangle < 1, \tag{8}$$

$$\langle \delta(\hat{Y}_3 - \hat{X}_2 - \hat{X}_4)^2 \rangle + \langle \delta(\hat{Y}_2 - \hat{X}_1 - \hat{X}_3)^2 \rangle < 1, \tag{9}$$

$$\langle \delta(\hat{Y}_3 - \hat{X}_2 - \hat{X}_4)^2 \rangle + \langle \delta(\hat{Y}_4 - \hat{X}_3)^2 \rangle < 1. \tag{10}$$

These inequalities make it possible to obtain a relationship between the quadratures of the cluster state and the quadratures of the squeezed oscillators on which this state was generated:

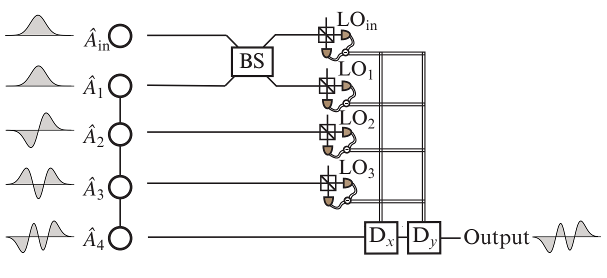

Figure 1. Schematic of one-way computing using a four-mode cluster state. The amplitudes $\hat{A}_1$–$\hat{A}_4$ have different mode profiles, but they are the amplitudes of the nodes of the cluster state; $\hat{A}_{\mathrm{in}}$ is the amplitude of the input state over which computations are performed; BS is the beam splitter; $\mathrm{LO}_{\mathrm{in}}$, LO_{1-3} are the local oscillators; and D_x and D_y perform photocurrent displacement operations.

$$\begin{aligned} \hat{X}_1 &= \frac{1}{\sqrt{2}}\hat{x}_1 + \frac{1}{\sqrt{10}}\hat{x}_2 + \sqrt{\frac{2}{5}}\hat{y}_3, \\ \hat{Y}_1 &= \frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_2 - \sqrt{\frac{2}{5}}\hat{x}_3, \end{aligned} \tag{11}$$

$$\begin{aligned} \hat{X}_2 &= -\frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_2 - \sqrt{\frac{2}{5}}\hat{x}_3, \\ \hat{Y}_2 &= \frac{1}{\sqrt{2}}\hat{x}_1 - \frac{1}{\sqrt{10}}\hat{x}_2 - \sqrt{\frac{2}{5}}\hat{y}_3, \end{aligned} \tag{12}$$

$$\begin{aligned} \hat{X}_3 &= \frac{1}{\sqrt{2}}\hat{y}_4 + \frac{1}{\sqrt{10}}\hat{y}_3 - \sqrt{\frac{2}{5}}\hat{x}_2, \\ \hat{Y}_3 &= -\frac{1}{\sqrt{2}}\hat{x}_4 - \frac{1}{\sqrt{10}}\hat{x}_3 - \sqrt{\frac{2}{5}}\hat{y}_2, \end{aligned} \tag{13}$$

$$\begin{aligned} \hat{X}_4 &= -\frac{1}{\sqrt{2}}\hat{x}_4 + \frac{1}{\sqrt{10}}\hat{x}_3 + \sqrt{\frac{2}{5}}\hat{y}_2, \\ \hat{Y}_4 &= -\frac{1}{\sqrt{2}}\hat{y}_4 + \frac{1}{\sqrt{10}}\hat{y}_3 - \sqrt{\frac{2}{5}}\hat{x}_2 \end{aligned} \tag{14}$$

The amplitude of the input field, over which we will carry out the calculations, is represented similarly to expansions (7):

$$\hat{A}_{\mathrm{in}}(t) = L_1(t)(\hat{X}_{\mathrm{in}} + \mathrm{i}\,\hat{Y}_{\mathrm{in}}) + \mathrm{vac}_{\mathrm{in}}. \tag{15}$$

The input signal profile must be consistent with the profile of the first cluster node. To perform calculations on the input state, we mix it to the cluster state. To do this, we mix the fields with the amplitudes $\hat{A}_{\mathrm{in}}(t)$ and $\hat{A}_1(t)$ on the beam splitter. The result of the transformation can be written in the form

$$\begin{aligned} \hat{A}'_{\mathrm{in}}(t) = \frac{L_1(t)}{\sqrt{2}}\Big[&\hat{X}_{\mathrm{in}} + \frac{1}{\sqrt{2}}\hat{x}_1 + \frac{1}{\sqrt{10}}\hat{x}_2 + \sqrt{\frac{2}{5}}\hat{y}_3 \\ &+ \mathrm{i}\Big(\hat{Y}_{\mathrm{in}} + \frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_2 - \sqrt{\frac{2}{5}}\hat{x}_3\Big)\Big] + \mathrm{vac}_{\mathrm{in}}, \end{aligned} \tag{16}$$

$$\begin{aligned} \hat{A}'_1(t) = \frac{L_1(t)}{\sqrt{2}}\Big[&\hat{X}_{\mathrm{in}} - \frac{1}{\sqrt{2}}\hat{x}_1 - \frac{1}{\sqrt{10}}\hat{x}_2 - \sqrt{\frac{2}{5}}\hat{y}_3 \\ &+ \mathrm{i}\Big(\hat{Y}_{\mathrm{in}} - \frac{1}{\sqrt{2}}\hat{y}_1 - \frac{1}{\sqrt{10}}\hat{y}_2 + \sqrt{\frac{2}{5}}\hat{x}_3\Big)\Big] + \mathrm{vac}_1. \end{aligned} \tag{17}$$

To perform the transformation over the input field $\hat{A}_{\mathrm{in}}$, it is necessary to measure the amplitudes $\hat{A}_{\mathrm{in}}(t)$, $\hat{A}'_1(t)$, $\hat{A}_2(t)$ and $\hat{A}_3(t)$ using four homodyne detectors whose local oscillators have amplitudes $\beta_k(t) = \beta_0 L_1(t)(\cos\theta_k + \mathrm{i}\sin\theta_k)$, where $k = \{\mathrm{in}, 1\}$, and $\beta_k(t) = \beta_0 L_k(t)(\cos\theta_k + \mathrm{i}\sin\theta_k)$, where $k = \{2, 3\}$, respectively. As a result, we obtain expressions for time-dependent photocurrent operators:

$$\begin{aligned} \sqrt{2}\,\hat{i}_{\mathrm{in}}(t) = \beta_0 L_1^2(t)\Big[&\Big(\hat{X}_{\mathrm{in}} + \frac{1}{\sqrt{2}}\hat{x}_1 + \frac{1}{\sqrt{10}}\hat{x}_2 + \sqrt{\frac{2}{5}}\hat{y}_3\Big)\cos\theta_{\mathrm{in}} \\ &+ \Big(\hat{Y}_{\mathrm{in}} + \frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_2 - \sqrt{\frac{2}{5}}\hat{x}_3\Big)\sin\theta_{\mathrm{in}}\Big], \end{aligned} \tag{18}$$

$$\begin{aligned} \sqrt{2}\,\hat{i}_1(t) = \beta_0 L_1^2(t)\Big[&\Big(\hat{X}_{\mathrm{in}} - \frac{1}{\sqrt{2}}\hat{x}_1 - \frac{1}{\sqrt{10}}\hat{x}_2 - \sqrt{\frac{2}{5}}\hat{y}_3\Big)\cos\theta_1 \\ &+ \Big(\hat{Y}_{\mathrm{in}} - \frac{1}{\sqrt{2}}\hat{y}_1 - \frac{1}{\sqrt{10}}\hat{y}_2 + \sqrt{\frac{2}{5}}\hat{x}_3\Big)\sin\theta_1\Big], \end{aligned} \tag{19}$$

$$\hat{i}_2(t) = \beta_0 L_2^2(t)\Big[\Big(-\frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_2 - \sqrt{\frac{2}{5}}\hat{x}_3\Big)\cos\theta_2 + \Big(\frac{1}{\sqrt{2}}\hat{x}_1 - \frac{1}{\sqrt{10}}\hat{x}_2 - \sqrt{\frac{2}{5}}\hat{y}_3\Big)\sin\theta_2\Big], \tag{20}$$

$$\hat{i}_3(t) = \beta_0 L_3^2(t)\Big[\Big(\frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_3 - \sqrt{\frac{2}{5}}\hat{x}_2\Big)\cos\theta_3 + \Big(-\frac{1}{\sqrt{2}}\hat{x}_4 - \frac{1}{\sqrt{10}}\hat{x}_3 - \sqrt{\frac{2}{5}}\hat{y}_2\Big)\sin\theta_3\Big], \tag{21}$$

which after integration with respect to time, taking into account the orthonormality of the set of profiles $\{L_i(t)\}$, lead us to the relations

$$\sqrt{2}\,\hat{i}_{\text{in}} = \beta_0\Big[\Big(\hat{X}_{\text{in}} + \frac{1}{\sqrt{2}}\hat{x}_1 + \frac{1}{\sqrt{10}}\hat{x}_2 + \sqrt{\frac{2}{5}}\hat{y}_3\Big)\cos\theta_{\text{in}} + \Big(\hat{Y}_{\text{in}} + \frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_2 - \sqrt{\frac{2}{5}}\hat{x}_3\Big)\sin\theta_{\text{in}}\Big], \tag{22}$$

$$\sqrt{2}\,\hat{i}_1 = \beta_0\Big[\Big(\hat{X}_{\text{in}} - \frac{1}{\sqrt{2}}\hat{x}_1 - \frac{1}{\sqrt{10}}\hat{x}_2 - \sqrt{\frac{2}{5}}\hat{y}_3\Big)\cos\theta_1 + \Big(\hat{Y}_{\text{in}} - \frac{1}{\sqrt{2}}\hat{y}_1 - \frac{1}{\sqrt{10}}\hat{y}_2 + \sqrt{\frac{2}{5}}\hat{x}_3\Big)\sin\theta_1\Big], \tag{23}$$

$$\hat{i}_2 = \beta_0\Big[\Big(-\frac{1}{\sqrt{2}}\hat{y}_1 + \frac{1}{\sqrt{10}}\hat{y}_2 - \sqrt{\frac{2}{5}}\hat{x}_3\Big)\cos\theta_2 + \Big(\frac{1}{\sqrt{2}}\hat{x}_1 - \frac{1}{\sqrt{10}}\hat{x}_2 - \sqrt{\frac{2}{5}}\hat{y}_3\Big)\sin\theta_2\Big], \tag{24}$$

$$\hat{i}_3 = \beta_0\Big[\Big(\frac{1}{\sqrt{2}}\hat{y}_4 + \frac{1}{\sqrt{10}}\hat{y}_3 - \sqrt{\frac{2}{5}}\hat{x}_2\Big)\cos\theta_3 + \Big(-\frac{1}{\sqrt{2}}\hat{x}_4 - \frac{1}{\sqrt{10}}\hat{x}_3 - \sqrt{\frac{2}{5}}\hat{y}_2\Big)\sin\theta_3\Big]. \tag{25}$$

We solve the resulting system of equations for the unknown quadratures $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$ and substitute them into equation (7) for the amplitude $\hat{A}_4(t)$, taking into account relationship (14). Thus, we find quadratures of the field $\hat{A}_4(t)$, which we denote by $\hat{X}_{\text{out}}(t)$ and $\hat{Y}_{\text{out}}(t)$. These quadratures are related to the quadratures of the initial signal $\hat{A}_{\text{in}}(t)$ as follows:

$$\begin{pmatrix}\hat{X}_{\text{out}}(t)\\ \hat{Y}_{\text{out}}(t)\end{pmatrix} = L_4(t)\,K(\theta_2,\theta_3)\,M(\theta_+,\theta_-)\begin{pmatrix}\hat{X}_{\text{in}}\\ \hat{Y}_{\text{in}}\end{pmatrix} + L_4(t)F\begin{pmatrix}\hat{y}_1\\ \hat{y}_2\\ \hat{y}_3\\ \hat{y}_4\end{pmatrix} + L_4(t)H\begin{pmatrix}\hat{i}_{\text{in}}\\ \hat{i}_1\\ \hat{i}_2\\ \hat{i}_3\end{pmatrix}, \tag{26}$$

where $\theta_\pm = \theta_1 \pm \theta_{\text{in}}$;

$$M(\theta_+,\theta_-) = \frac{1}{\sin\theta_-}\begin{pmatrix}\cos\theta_+ + \cos\theta_- & \sin\theta_+\\ -\sin\theta_+ & \cos\theta_+ - \cos\theta_-\end{pmatrix} \tag{27}$$

and

$$K(\theta_2,\theta_3) = \begin{pmatrix}\dfrac{\cos(\theta_2+\theta_3)}{\sin\theta_2\sin\theta_3} & \cot\theta_3\\ -\cot\theta_2 & -1\end{pmatrix} \tag{28}$$

are the matrices; and F and H are also matrices depending on $\theta_{\text{in}}, \theta_1, \theta_2$, and θ_3. The general form of the quadratures $\hat{X}_{\text{out}}(t)$ and $\hat{Y}_{\text{out}}(t)$ is given in the Appendix.

Equation (26) consists of three terms. The first term corresponds to the desired transformation over the quadratures of the field $\hat{A}_{\text{in}}$. This transformation is completely determined by the choice of the angles θ_{in}, θ_1, θ_2, and θ_3 for homodyne detection. The second term contains only squeezed quadratures with numerical coefficients, and the last one contains photocurrent operators. In the computations on cluster states, we measure the photocurrents; therefore, we must pass from the operators $\hat{i}_{\text{in}}$, $\hat{i}_1$, $\hat{i}_2$, and $\hat{i}_3$ to the corresponding measured c-number values. Then in the last term on the right-hand side of (26) there will be only classical quantities that can be compensated for by shifting the quadratures of the output field. Formally, the displacement is described by the action of the operators $\hat{D}_x(s) = \exp(-\mathrm{i}s\hat{Y})$ and $\hat{D}_y(s) = \exp(-\mathrm{i}s\hat{X})$:

$$\hat{D}_x^\dagger(s(t))\,\hat{X}(t)\,\hat{D}_x(s(t)) = \hat{X}(t) + s(t),\qquad \hat{D}_x^\dagger(s(t))\,\hat{Y}(t)\,\hat{D}_x(s(t)) = \hat{Y}(t), \tag{29}$$

$$\hat{D}_y^\dagger(s(t))\,\hat{Y}(t)\,\hat{D}_y(s(t)) = \hat{Y}(t) + s(t),\qquad \hat{D}_y^\dagger(s(t))\,\hat{X}(t)\,\hat{D}_y(s(t)) = \hat{X}(t). \tag{30}$$

It is worth noting that the displacement value is selected using the feed-forward procedure in each individual experiment. The essence of this operation is that we send the measurement results to the physical devices that perform the transformations $\hat{D}_x(s)$ and $\hat{D}_y(s')$ before the field arrives at them. Thus, we prepare these devices so that the quadratures of the light that has come on them are displaced in the way we need. If in our problem we apply transformations to output quadratures,

$$\hat{D}_x(-L_4(t)(H_{11}i_{\text{in}} + H_{12}i_1 + H_{13}i_2 + H_{14}i_3)),$$

$$\hat{D}_y(-L_4(t)(H_{21}i_{\text{in}} + H_{22}i_1 + H_{23}i_2 + H_{24}i_3)),$$

then we completely compensate for the last term in expression (26).

The second term on the right-hand side of (26) is proportional to the squeezed quadratures $\hat{y}_1$, $\hat{y}_2$, $\hat{y}_3$ and $\hat{y}_4$. Since each computation terminates by the procedure for measuring the resulting state, these terms under sufficiently good initial squeezing can be neglected as small corrections. Therefore, the final equation relating the input and output quadratures takes the form

$$\begin{pmatrix}\hat{X}_{\text{out}}(t)\\ \hat{Y}_{\text{out}}(t)\end{pmatrix} = L_4(t)\,K(\theta_2,\theta_3)\,M(\theta_+,\theta_-)\begin{pmatrix}\hat{X}_{\text{in}}\\ \hat{Y}_{\text{in}}\end{pmatrix}. \tag{31}$$

This equation demonstrates a linear relationship between the input and output quadratures. Hence, we can use the Bloch–Messiah reduction theorem [21], which states that the matrix $K(\theta_2,\theta_3)M(\theta_+,\theta_-)$ relating the quadratures can be expanded as follows:

$$K(\theta_2,\theta_3)\,M(\theta_+,\theta_-) = R(\phi_1)\,S(r)\,R(\phi_2), \tag{32}$$

where the matrices of rotation (R) and squeezing (S) have form (6), and the rotation angles ϕ_1 and ϕ_2 and the squeezing parameter r are explicitly expressed in terms of the homodyne angles $\theta_{\rm in}$, θ_1, θ_2, and θ_3. As was shown in [27], this is the most common expansion, i.e., by choosing the angles of homodynes, we can perform an arbitrary linear transformation over the input state of the field.

The quadratures of the cluster state node (26) after the reduction procedure associated with measuring the quadrature of the previous node are proportional to the mode profile of this node. In other cases, expression (26) completely coincides with the analogous formula without taking into account the specificity of the various profiles of cluster nodes [27]. As we see, there are no additional noise terms associated with the mismatch of mode profiles. Thus, the computation procedure proves to be stable with respect to the change in the mode composition of the cluster nodes. Note that the signal profile over which the computation is performed will change in this case and will coincide with the profile of the last measured cluster node. A completely analogous situation arises when two-qubit operations are performed both on linear clusters and on clusters with a more complex topology.

4. Conclusions

We have shown that the formal application of the unitary transformation, often used in describing the construction of a cluster from field oscillators with mismatched amplitude profiles, leads to the addition of noise if such a transformation is implemented by means of standard linear optics devices. To perform this transformation, it is necessary to use an additional device that modifies the profile of the field amplitude and maintains the quantum correlations in it. As a possible variant of such a converter, one can consider a quantum memory cell [24, 25]. At the same time, if it is possible to generate a cluster whose node amplitudes have different time profiles, this feature does not interfere with the performance of computations on the cluster.

It should be noted that the simplicity of constructing a cluster state from field oscillators in a squeezed quantum state is considered to be a great advantage of these many-particle-entangled systems. Excessive noise in them is often estimated based on the number of elements that perform mixing and rotation of field amplitudes. These estimates cease to be true if it is assumed that such nontrivial devices as quantum converters of the signal profile are embedded in the scheme.

We have conducted a discussion, assuming that the field modes differ in temporal profiles. However, the same reasoning is valid if the field modes are obtained by any other expansion with respect to a complete orthonormal set of functions. When mixing such modes, it is necessary to take into account the presence of orthogonal modes in the vacuum state. Note that, although for time modes the mechanisms of their quantum transformation have already been proposed, the development of devices that perform a noiseless conversion of other degrees of freedom requires discussion.

We would like to emphasise that in considering the formation of many-particle-entangled quantum systems, the issue of mixing different modes should not remain only a mathematical operation, but requires specifying the used mixing procedure.

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Appendix. Relationship between input and output quadratures of the oscillators in quantum computations

Let us expand the explicit form of the quadratures $\hat{X}_{\rm out}$ and $\hat{Y}_{\rm out}$ in expression (26):

$$\begin{aligned}\hat{X}_{\rm out}(t) &= \frac{L_4(t)}{\sin\theta}\\ &\quad\times\bigg\{\bigg[\frac{\cos(\theta_2+\theta_3)(\cos\theta_+ + \cos\theta_-)}{\sin\theta_2\sin\theta_3} - \cot\theta_3\sin\theta_+\bigg]\hat{x}_{\rm in}\\ &\quad+\bigg[\frac{\cos(\theta_2+\theta_3)\sin\theta_+}{\sin\theta_2\sin\theta_3} + \cot\theta_3(\cos\theta_+ - \cos\theta_-)\bigg]\hat{y}_{\rm in}\bigg\}\\ &\quad+L_4(t)[(1-2\cot\theta_2\cot\theta_3)\hat{y}_1 + 3(\cot\theta_3)\hat{y}_2\\ &\quad+(-2-\cot\theta_2\cot\theta_3)\hat{y}_3 + (\cot\theta_3)\hat{y}_4]\\ &+\frac{L_4(t)}{\sin\theta_-}\Big[(\cos\theta_1 - \cos\theta_1\cot\theta_2\cos\theta_3 + \cot\theta_3\sin\theta_1)\hat{i}_{\rm in}\\ &+(\cos\theta_{\rm in} - \cos\theta_{\rm in}\cot\theta_2\cot\theta_3 + \cot\theta_3\sin\theta_{\rm in})\hat{i}_1\\ &\quad-\frac{\cot\theta_3}{\sin\theta_2}\hat{i}_2 - \frac{1}{\sin\theta_3}\hat{i}_3\Big],\end{aligned} \tag{A1}$$

$$\begin{aligned}\hat{Y}_{\rm out}(t) &= \frac{L_4(t)}{\sin\theta_-}\{[-\cot\theta_2(\cos\theta_+ + \cos\theta_-) + \sin\theta_+]\hat{x}_{\rm in}\\ &\quad+(-\cot\theta_2\sin\theta_+ - \cos\theta_+ + \cos\theta_-)\hat{y}_{\rm in}\}\\ &\quad+L_4(t)[2(\cot\theta_2)\hat{y}_1 - 2\hat{y}_2 + (\cot\theta_2)\hat{y}_3 + \hat{y}_4]\\ &\quad+\frac{L_4(t)}{\sin\theta_2}\Big[\frac{\cos(\theta_{\rm in}+\theta_2)}{\sin\theta_-}\hat{i}_{\rm in} + \frac{\cos(\theta_1+\theta_2)}{\sin\theta_-}\hat{i}_1 - \hat{i}_2\Big].\end{aligned} \tag{A2}$$

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