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Energy density in a collapsing electromagnetic wave

I.A. Artyukov, A.V. Vinogradov, N.V. D'yachkov, R.M. Feshchenko

Abstract. **The relationship between the energy and the width of the spectrum of a collapsing electromagnetic wave and the maximum energy density at the focal point has been found. The solution method can be used to simulate the propagation of ultrashort pulses without using the slowly varying amplitude approximation.**

Keywords: *collapsing electromagnetic wave, energy density, ultrashort pulses.*

1. Introduction

Along with numerical methods for solving Maxwell's equations in a vacuum [1, 2], analytical methods continue to be widely used to simulate light beams. In this case, we deal with approximate methods (the method of slowly varying amplitudes, the Kirchhoff method, etc.), or with exact solutions of approximate equations for electromagnetic waves*. Such solutions include the Fresnel integral and its generalisation [3, 4], Gauss [5], Bessel [6], spiral and other beams [7].

In this paper, we consider the class of exact solutions of Maxwell's equations. We make use of these equations to investigate collapsing electromagnetic beams. An important property of these solutions is the finiteness of the total energy of the field. From previous papers on this topic, we should note papers [8, 9] and especially Refs [10–12].

2. Solution of Maxwell's equations in the form of travelling spherical waves

Maxwell's equations in vacuum allow calibration of potentials in the form [13]

$$\mathrm{div}\,\boldsymbol{A} = 0, \quad \phi = 0 \tag{1}$$

and are reduced in this case to the wave equation for the vector potential $\boldsymbol{A}$, through which the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{H}$) fields are expressed:

$$\Delta\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = 0, \quad \boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}, \quad \boldsymbol{H} = \mathrm{rot}\,\boldsymbol{A}. \tag{2}$$

It is easy to verify that equation (2) with calibration (1) satisfies the potential of the form

$$\boldsymbol{A} = \boldsymbol{l}\times\mathrm{grad}\,u \tag{3}$$

(where $\boldsymbol{l}$ is a unit axial vector that is independent of time and coordinates) provided that the function $u(\boldsymbol{r}, t)$ also satisfies wave equation (2). Its spherically symmetric solution $u(r, t)$ is made up of two opposite waves, described by arbitrary functions f_+ and f_-:

$$u = \frac{f_+(r - ct) + f_-(r + ct)}{r}, \tag{4}$$

and the solution without singularity at zero has the form [14]

$$u = \frac{f(ct + r) - f(ct - r)}{r}, \tag{5}$$

where $f(s)$ is also an arbitrary function. If it decreases at large $|s|$ quickly enough, then the electromagnetic pulse (or beam) has finite energy.

Using (2) and vector potential (3), (5), it is easy to obtain expressions for the fields $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} = -\frac{1}{c}\frac{\partial^2 u}{\partial t\partial r}\boldsymbol{l}\times\boldsymbol{n}, \tag{6}$$

$$\boldsymbol{H} = \mathrm{rot}\,\boldsymbol{A} = \Big(\Delta u - \frac{1}{r}\frac{\partial u}{\partial r}\Big)\boldsymbol{l} - \Big(r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial u}{\partial r}\Big)(\boldsymbol{l}\boldsymbol{n})\boldsymbol{n}, \tag{7}$$

where

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big);$$

and $\boldsymbol{n}$ is the unit vector along the radius vector $\boldsymbol{r}$. Formulae (5)–(7), containing an arbitrary function f, describe a certain type of light beams with a finite energy that satisfy Maxwell's equations in vacuum.

3. Total pulse energy, spectrum and energy density

First, we express these quantities in terms of the function $f(s)$, which determines fields (6) and (7).

3.1. Pulse energy

The total energy $\mathcal{E}$ of a pulse (5)–(7) is found by the formula

*The most known are the scalar wave equation (Helmholtz equation) and the parabolic wave equation.

I.A. Artyukov, A.V. Vinogradov, R.M. Feshchenko P.N. Lebedev Physical Institute, Russian Academy of Sciences, Leninsky prosp. 53, 119991 Moscow, Russia; e-mail: vinograd@sci.lebedev.ru;
N.V. D'yachkov P.N. Lebedev Physical Institute, Russian Academy of Sciences, Leninsky prosp. 53, 119991 Moscow, Russia; Moscow Institute of Physics and Technology (State University), Institutskiy per., 9, 141701 Dolgoprudny, Moscow region

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$$\mathcal{E} = \lim_{r\to\infty}\int_0^\infty \Phi(r,t)\,\mathrm{d}t, \tag{8}$$

where

$$\Phi(r,t) = r^2\iint I(\boldsymbol{r},t)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{9a}$$

is the energy flux through a sphere of radius r; θ and φ are the angles in the spherical coordinate system; and

$$I(\boldsymbol{r},t) = \frac{c}{4\pi}\boldsymbol{n}[\boldsymbol{E}\times\boldsymbol{H}] \tag{9b}$$

is the radial intensity. Substituting field strengths (6) and (7) in (9b) and then in (9a), we first determine the energy flux:

$$\Phi(r,t) = -r^2\frac{2}{3}\frac{\partial v}{\partial t}\left(\frac{v}{r}+\frac{\partial v}{\partial r}\right),\quad v = \frac{\partial u}{\partial r}, \tag{10}$$

and then, following (8), the total pulse energy:

$$\mathcal{E} = \frac{2}{3}\int_{-\infty}^{\infty} g^2(s)\,\mathrm{d}s,\quad g(s) = f''(s). \tag{11}$$

3.2. Pulse spectrum in the far zone

Let us return to formula (9b) for the radial intensity, directing the z axis along the vector $\boldsymbol{l}$. Substituting into it the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ from (5)–(7), we obtain

$$I(\boldsymbol{r},t) = \frac{\sin^2\theta}{4\pi}\left(\frac{1}{r}\frac{\partial u}{\partial r} - \Delta u\right)\frac{\partial^2 u}{\partial t\,\partial r}. \tag{12}$$

The integral of intensity (12) over time, similar to (8), for large r gives fluence (the energy of the radiation incident on the unit area)

$$F(\boldsymbol{r}) = \lim_{r\to\infty}\int_0^\infty I(\boldsymbol{r},t)\,\mathrm{d}t. \tag{13}$$

Using approximate expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ for large r, formula (13) can be easily simplified, as is expression (8) for the total energy. Then, similar to (11), fluence (13) takes the form

$$F(\boldsymbol{r}) \approx \frac{\sin^2\theta}{4\pi r^2}\int_{-\infty}^{\infty} g^2(s)\,\mathrm{d}s. \tag{14}$$

This result is expected, since the total energy $\mathcal{E}$ of the pulse is actually in the numerator. From (14) it is easy to find the spectral power density:

$$F_\omega(\boldsymbol{r}) \approx \frac{\sin^2\theta}{c^2r^2}|g(k)|^2,\quad k = \omega/c,$$
$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\exp(\mathrm{i}ks)\,\mathrm{d}s,\quad F(\boldsymbol{r}) = \int_0^\infty F_\omega(\boldsymbol{r})\,\mathrm{d}\omega. \tag{15}$$

We take into account that $g(s)$ is a real function, and $|g(k)|^2$ is even. It is natural to identify the quantity $F_\omega(\boldsymbol{r})$ with the pulse spectrum in the far zone.

3.3. Energy density

The energy density ϵ in the centre of the pulse is determined by the formula

$$\epsilon = \frac{\boldsymbol{E}^2+\boldsymbol{H}^2}{8\pi}\bigg|_{r=0}. \tag{16}$$

Hence, using the general formulas for the fields (5)–(7), we find

$$\boldsymbol{E}(ct) = 0,\quad \boldsymbol{H}(ct) = \frac{4}{3}f'''(ct)\ \text{ and }\ \epsilon(ct) = \frac{2}{9\pi}[g'(ct)]^2. \tag{17}$$

From (17) it is seen that the density of the electromagnetic energy $\epsilon(t)$ at the origin of coordinates increases with time from $t = -\infty$ and then decreases to zero.

We emphasize that expression (17) for the energy density $\epsilon(ct)$ at the centre of a collapsing pulse, as well as expression (11) for the total energy $\mathcal{E}$, is valid for any function $f(s)$ determining the spatiotemporal pulse shape [see (4)]. The proof of the validity of formulas (11) and (17) will be given in our next paper.

4. Collapse of a quasi-monochromatic wave

Let us select the pulse shape (5)–(7) by setting the second derivative $f(s)$ in the form

$$f''(s) = g(s) = C_1\exp(-s^2/a^2)\sin(qs), \tag{18}$$

where the normalisation coefficient C_1, in accordance with formula (11) for the total energy of the pulse, is determined by the expression

$$C_1^2 = 6\mathcal{E}\{a\sqrt{2\pi}[1-\exp(-q^2a^2/2)]\}^{-1}; \tag{19}$$

and a and q are the pulse length and the wave number, respectively. Following Section 3, we consider the spectrum and energy density of such a pulse in focus. Substituting (18) into (15), we find the fluence spectral function:

$$F_\omega(\boldsymbol{r}) = \frac{\sin^2\theta}{c^2r^2}g^2(k) = \frac{a^2C_1^2\sin^2\theta}{16\pi c^2r^2}$$
$$\times\left\{\exp\left[-\frac{(\omega-cq)^2a^2}{4c^2}\right] - \exp\left[-\frac{(\omega+cq)^2a^2}{4c^2}\right]\right\}^2. \tag{20}$$

Using this function, we can easily determine the centre frequency ω_0 and the width $\delta\omega$ of the emission spectrum. As was to be expected, in the case of a quasi-monochromatic wave, when $aq \gg 1$, we have

$$\omega_0 = cq,\quad \delta\omega = c/a. \tag{21}$$

The energy density in the centre is determined by formula (17), which shows that for pulse (18) the maximum value of ϵ_m is reached at $t = 0$:

$$\epsilon_\mathrm{m} = \frac{1}{3}\left(\frac{2}{\pi}\right)^{3/2}\frac{\mathcal{E}q^2}{a[1-\exp(-q^2a^2/2)]}. \tag{22}$$

In the case of a quasi-monochromatic pulse, when $aq \gg 1$ and expression (21) is valid, formula (22) implies the expected (up to a factor) dependence of the energy density at the centre on the total pulse energy, wavelength and spectrum width:

$$\epsilon_\mathrm{m} = \frac{8}{3}(2\pi)^{3/2}\frac{\mathcal{E}}{\lambda_0^3}\frac{\delta\omega}{\omega_0},\quad \lambda_0 = \frac{2\pi c}{\omega_0},\quad aq = \frac{\omega_0}{\delta\omega} \gg 1. \tag{23}$$

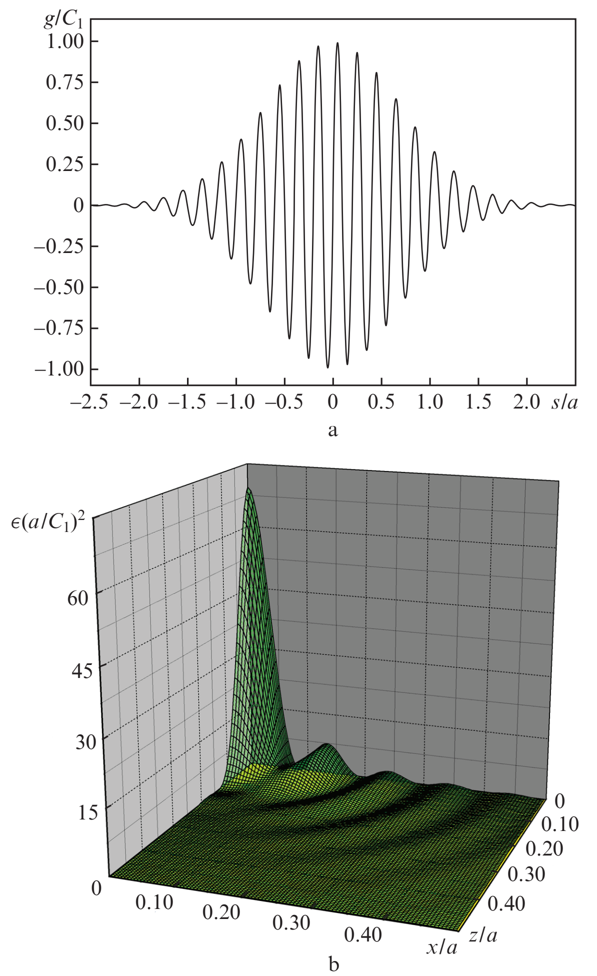

Figure 1. (a) Pulse shape $g(s)$ given by formula (18) with $qa = 10\pi$ and (b) distribution of the energy density $\epsilon(x, z)$ in the vicinity of the origin at $t = 0$.

The two-dimensional distribution of the energy density $\epsilon(\boldsymbol{r})$ at the moment of the pulse collapse (Fig. 1a) is shown in Fig. 1b.

5. Discussion

Using a model pulse, which is an exact solution of the equations for the electromagnetic field, we have studied the solution that represents a collapsing and then expanding shell with fast field oscillations. The relation between the energy density ϵ_{m} at the focal point and the total energy of the pulse $\mathcal{E}$, the wavelength λ_0 and the spectrum width $\delta\omega$ is obtained. The question of the generality of result (22), (23) remains open until other, more realistic, exact solutions of equations for a field with a finite energy are investigated. In particular, they can be constructed by shifting field (5) to the complex plane along one of the coordinates [9, 11].

Despite the obvious difficulties in implementing the spatio-temporal structure of field (5)–(7) with high symmetry, it seems to be interesting to compare in detail relation (23) with the energy density attainable on existing and projected relativistic intensity laser facilities.

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