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Angular momentum of elliptically polarised cnoidal waves and breathers in a nonlinear gyrotropic medium with frequency dispersion

V.A. Makarov, V.M. Petnikova

Abstract. **We have analysed expressions for the spin part of the density vector of the angular momentum (moment of momentum) of elliptically polarised cnoidal waves and breathers propagating in an isotropic gyrotropic medium with second order frequency dispersion and spatial cubic nonlinearity dispersion. We have found that, due to nonlinear gyration, the density of the angular momentum depends on time and propagation coordinates.**

Keywords: *angular momentum of the electromagnetic field, cubic nonlinearity, gyrotropy, optically active medium, cnoidal waves, breathers.*

1. Introduction

Such fundamental concepts of electrodynamics as the energy–momentum tensor and the angular momentum of an electromagnetic wave are still being actively discussed [1–6]. In the paraxial approximation, the latter is traditionally represented as a sum of orbital and spin parts. For a plane electromagnetic monochromatic wave propagating along the z axis, the orbital part of the angular momentum is identically zero, and the density vector $\boldsymbol{J}$ of the spin part of the angular momentum related to its polarisation is directed along the z axis. Its projection J on this axis is associated with slowly varying amplitudes $\mathcal{E}_{\pm} = \mathcal{E}_x \pm \mathrm{i}\mathcal{E}_y$ of circularly polarised components of the electric field [7–10]:

$$J = (|\mathcal{E}_-|^2 - |\mathcal{E}_+|^2)/(16\pi k). \quad (1)$$

A similar result was obtained during quantisation of an electromagnetic field; the spin part of its angular momentum is equal to $\hbar\sum_{\boldsymbol{k}}(n_{\boldsymbol{k},-} - n_{\boldsymbol{k},+})$ [7–10]. Here, $\hbar$ is the Planck constant, and $n_{\boldsymbol{k},\pm}$ is the number of right- and left-handed polarised photons in the mode determined by the wave vector $\boldsymbol{k}$. In both cases we are dealing with real experiments, when an abstract plane wave is limited to an arbitrarily large, but finite aperture and duration of exposure; in quantising the field, this is due to the finiteness of the volume during normalisation of the wave functions [3]. The J value is proportional to one of the Stokes parameters [11, 12] and is related to such fundamental photon characteristics as helicity and spin [8, 10, 12]. The physical meaning of the orbital and spin parts of the angular momentum and the relationship between them and similar terms in the quantum-mechanical consideration of the propagation of light are interpreted ambiguously and for more than forty years have been the subject of many discussions that are supported by a large number of recent works. They analyse the methods for obtaining laser beams with different angular momenta and investigate the transformations of their orbital and spin parts in the process of light propagation and its interaction with different objects.

Beams with different angular momenta are widely used to change the orientation and rotation of microparticles trapped by light (optical 'tweezers'), to control mechanical micromachines [13], as well as in optical calculations, during information transfer, etc. To date, there exist many methods for producing such beams, including the use of a spiral phase plate [14], a special diffractive element (holographic pattern) [15], and a system of cylindrical lenses [16]. Optical elements have been developed, with the help of which the mutual transformation of the orbital and spin parts of the angular momentum of the propagating wave is carried out [17].

However, in the numerical and analytical solution of most problems of nonlinear optics, almost always the fulfillment of the law of energy conservation, more rarely the law of momentum conservation, and almost never the law of angular momentum conservation are controlled. The study of the transformations of the 'orbital' and 'spin' parts of laser pulse angular momenta in the process of their interaction in nonlinear media within the framework of classical electrodynamics not only contributes to the development of nonlinear optics, but also contributes to a deeper understanding of the fundamental laws of nature. Recall that the Planck constant was measured in the classical Beth experiment [4, 18], which is now actually repeated in [18] for polarised waves propagating in a birefringent crystal. In this work, the spin angular momentum was measured and the possibility of using the obtained results in integrated optics and optomechanical devices was demonstrated. This explains our interest in the analysis of the spin part of the angular momentum of elliptically polarised plane waves propagating in an isotropic gyrotropic medium with spatial dispersion of cubic nonlinearity and second order frequency dispersion of group velocities. Elliptically polarised cnoidal waves [19–21] and vector breathers [22] can propagate in such a medium. Below, we will show that the flux density of the spin part of the angular momentum of such waves can obviously depend on both the running time and the propagation coordinate. This may provide new material for discussing issues related to the angular momentum of propagating waves.

V.A. Makarov, V.M. Petnikova Faculty of Physics and International Laser Center, M.V. Lomonosov Moscow State University, Vorob'evy gory, 119991 Moscow, Russia; e-mail: vamakarov@phys.msu.ru

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2. Spin angular momentum of elliptically polarised cnoidal waves and breathers

The variation of slowly varying amplitudes of circularly polarised components of elliptically polarised cnoidal waves and breathers propagating in a nonlinear isotropic gyrotropic medium with second order dispersion of group velocities ($\partial^2 k/\partial\omega^2 = k_2 \neq 0$) is described by a nonintegrable system of partial differential equations [19–22]:

$$\frac{\partial}{\partial z}\mathcal{E}_{\pm} - \mathrm{i}\frac{k_2}{2}\frac{\partial^2}{\partial t^2}\mathcal{E}_{\pm} + \mathrm{i}\Big[\mp\rho_0 + \Big(\frac{\sigma_1}{2}\mp\rho_1\Big)\Big]|\mathcal{E}_{\pm}|^2$$

$$+\Big(\frac{\sigma_1}{2}+\sigma_2\Big)|\mathcal{E}_{\mp}{}^2\Big]\mathcal{E}_{\pm} = 0. \quad (2)$$

Here, ω is the frequency of the propagating wave; k is the modulus of its wave vector directed along the z axis; t is the running time (in own coordinate system running with group velocity); and $\sigma_1 = 4\pi\omega^2\chi^{(3)}_{xyxy}/(kc^2)$ and $\sigma_2 = 2\pi\omega^2\times\chi^{(3)}_{xxyy}/(kc^2)$ are related to the independent components of the local cubic nonlinearity tensor $\hat{\chi}^{(3)}(\omega; -\omega,\omega,\omega)$, and $\rho_{0,1} = 2\pi\omega^2\gamma_{0,1}/c^2$ are defined through the pseudoscalar constants $\gamma_{0,1}$ of linear and nonlinear gyration.

The particular periodic solutions to system (2) in the form of cnoidal waves, satisfying the condition of linear coupling of the intensities of the right- (plus) and left-handed (minus) polarised components,

$$\delta_+|\mathcal{E}_+|^2 + \delta_-|\mathcal{E}_-|^2 = \delta_0, \quad (3)$$

($\delta_{0,\pm}$ are constants) can be divided into two groups [19–21]. The first includes nine solutions of $\mathcal{E}_{\pm}$, the phases of which depend only on z, and the amplitudes $\mathcal{E}_{\pm}(t)$ are expressed through all possible pairwise combinations of Jacobi elliptic functions, $\mathrm{sn}(\nu t,\mu)$, $\mathrm{cn}(\nu t,\mu)$ and $\mathrm{dn}(\nu t,\mu)$ [23]. The modulus of elliptic functions μ and the scale factor ν satisfy the inequalities $0 \leqslant \mu \leqslant 1$, $\nu > 0$, and are free parameters. The domains of existence of these solutions are determined from the condition $|\mathcal{E}_{\pm}|^2 \geqslant 0$.

Projections of the density vector of the spin part of the angular momentum of the waves with the components $|\mathcal{E}_{\pm}(t)|^2 = -(\mu\nu)^2 k_2(\sigma_2\mp\rho_1)\mathrm{cn}^2(\nu t,\mu)/\beta$ and $|\mathcal{E}_{\mp}(t)|^2 = -\nu^2\times k_2(\sigma_2\pm\rho_1)\mathrm{dn}^2(\nu t,\mu)/\beta$ are easily found by substituting these solutions into (1):

$$J_{\mathrm{c}_{\pm}\mathrm{d}_{\mp}} = \frac{\mp\nu^2 k_2[(\sigma_2\pm\rho_1) - \mu^2(\sigma_2\mp\rho_1)\mp 2\mu^2\rho_1\mathrm{sn}^2(\nu t,\mu)]}{16\pi k\beta}. \quad (4)$$

Here, $\beta = \rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2$; the subscripts $\mathrm{c}_{\pm},\mathrm{d}_{\pm}$ (hereafter, $\mathrm{s}_{\pm}$) correspond to the first letters of the elliptic functions and the corresponding polarisations of the components. Such waves exist if

$$\sigma_2 > 0,\ k_2\beta < 0,\ \rho_1 < |\sigma_2|$$

or (5)

$$\sigma_2 < 0,\ k_2\beta > 0,\ \rho_1 < |\sigma_2|.$$

Firstly, waves with components $|\mathcal{E}_{\pm}(t)|^2 = (\mu\nu)^2 k_2(\sigma_2\mp\rho_1)\times\mathrm{sn}^2(\nu t,\mu)/\beta$ and $|\mathcal{E}_{\mp}(t)|^2 = -(\mu\nu)^2 k_2(\sigma_2\pm\rho_1)\mathrm{cn}^2(\nu t,\mu)/\beta$ can propagate in a medium. The projections of the density vector of the spin part of the angular momentum, $J_{\mathrm{s}_{\pm}\mathrm{c}_{\mp}}$, of such waves are given by the expressions

$$J_{\mathrm{s}_{\pm}\mathrm{c}_{\mp}}(t) = \frac{\mp\mu^2\nu^2 k_2[(\sigma_2\pm\rho_1)\mp 2\rho_1\mathrm{sn}^2(\nu t,\mu)]}{16\pi k\beta}. \quad (6)$$

Secondly, waves with $|\mathcal{E}_{\pm}(t)|^2 = (\mu\nu)^2 k_2(\sigma_2\mp\rho_1)\mathrm{sn}^2(\nu t,\mu)/\beta$ and $|\mathcal{E}_{\mp}(t)|^2 = -\nu^2 k_2(\sigma_2\pm\rho_1)\mathrm{dn}^2(\nu t,\mu)/\beta$ can also propagate. The projections of the density vector of the spin part of the angular momentum, $J_{\mathrm{s}_{\pm}\mathrm{d}_{\mp}}$ have the form:

$$J_{\mathrm{s}_{\pm}\mathrm{d}_{\mp}}(t) = \frac{\mp\nu^2 k_2[(\sigma_2\pm\rho_1)\mp 2\mu^2\rho_1\mathrm{sn}^2(\nu t,\mu)]}{16\pi k\beta}. \quad (7)$$

The domain of existence of such solutions is determined by different inequalities. For the upper sign,

$$\rho_1 > 0,\ k_2\beta < 0,\ \sigma_2 < |\rho_1|, \quad (8)$$

and for the lower sign,

$$\rho_1 < 0,\ k_2\beta > 0,\ \sigma_2 < |\rho_1|. \quad (9)$$

The first group also includes three degenerate waves, whose amplitudes contain the same elliptic functions:

$$|\mathcal{E}_{\pm}(t)|^2 = (\mu\nu)^2 k_2(\sigma_2\mp\rho_1)\mathrm{sn}^2(\nu t,\mu)/\beta, \quad (10)$$

$$|\mathcal{E}_{\pm}(t)|^2 = -(\mu\nu)^2 k_2(\sigma_2\mp\rho_1)\mathrm{cn}^2(\nu t,\mu)/\beta, \quad (11)$$

$$|\mathcal{E}_{\pm}(t)|^2 = -\nu^2 k_2(\sigma_2\mp\rho_1)\mathrm{dn}^2(\nu t,\mu)/\beta. \quad (12)$$

The projections of the density vector of the spin part of the angular momentum of these waves are given by the formulae

$$J_{\mathrm{s,c,d}}(t) = \frac{\rho_1\nu^2 k_2}{8\pi k\beta}f_{\mathrm{s,c,d}}. \quad (13)$$

In (13), $f_{\mathrm{s}} = \mu^2\mathrm{sn}^2(\nu t,\mu)$, $f_{\mathrm{c}} = \mu^2[\mathrm{sn}^2(\nu t,\mu) - 1]$, and $f_{\mathrm{d}} = \mu^2\times\mathrm{sn}^2(\nu t,\mu) - 1$. The domain of existence of solution (10) is determined by the inequalities

$$\sigma_2 > 0,\ k_2\beta > 0,\ \rho_1 < |\sigma_2|$$

or (14)

$$\sigma_2 < 0,\ k_2\beta < 0,\ \rho_1 < |\sigma_2|,$$

whereas the domain of existence of solutions (11) and (12) coincide with (5). Note that for $\rho_1 = 0$, the right-hand side of (13) vanishes.

It can be seen from (4), (6), (7) and (13) that the dependence of all projections of the density vector of the spin part of the angular momentum on the running time is caused by the nonlocality of the nonlinear optical response of the gyrotropic medium and has different signs for different solutions (for $\rho_1 = 0$ the dependence on running time disappears). However, the frequency of oscillations of elliptic functions for nonlinear optically active media is high enough to produce real sample oscillations. The time-averaged contribution of these oscillations is proportional to

$$\langle\mathrm{sn}^2(\nu t,\mu)\rangle_t = \lim_{T\to\infty}(T)^{-1}\int_0^T\mathrm{sn}^2(\nu t,\mu)\mathrm{d}t$$

$$= \mu^{-2}[1 - \mathrm{E}(\mu)/\mathrm{K}(\mu)]. \quad (15)$$

Here, $K(\mu)$ and $E(\mu)$ are complete elliptic integrals of the first and second kind. Note that for $\mu = 1$ we have $\mathrm{sn}(vt, 1) = \mathrm{th}(vt)$, and $\langle \mathrm{sn}^2(vt, 1)\rangle_t = 1$.

In the second group of solutions to system (2), the amplitudes $|\mathcal{E}_\pm(t)|$ of circularly polarised components are expressed in terms of the Jacobi elliptical sine, and their phases $\varphi_\pm(z,t) = \mathrm{Arg}\{\mathcal{E}_\pm(t)\}$ explicitly depend on the coordinate and on the running time. In this case, the phase derivative determines the frequency shift of the components, i.e., their chirp [21]. The squares of the amplitude moduli of such solutions are expressed as

$$|\mathcal{E}_\pm(t)|^2 = |\mathcal{E}_\pm(0)|^2 [1 + m_\pm \mathrm{sn}^2(\tilde{v}t,\tilde{\mu})], \tag{16}$$

where $m_\pm = \tilde{\mu}^2\tilde{v}^2 k_2(\sigma_2 \mp \rho_1)/[|\mathcal{E}_\pm(0)|^2\beta]$; $\tilde{v}$ and $\tilde{\mu}$ are determined by the initial chirp values

$$\left(\frac{\mathrm{d}\varphi_\pm}{\mathrm{d}t}\right)_{t=0} = \pm\tilde{v}\sqrt{\frac{(\tilde{\mu}^2 + m_\pm)(1 + m_\pm)}{m_\pm}}. \tag{17}$$

The requirements of the positivity of the right-hand side of (16) and the radicand expression on the right-hand side of (17) impose certain restrictions on the values of $\tilde{v}$ and $\tilde{\mu}$ [21]. Those solutions of this group, in which the intensities of both circularly polarised field components at the point $t = 0$ begin to decrease, exist when conditions (5) are fulfilled. If one component at the initial time $t = 0$ increases and the other decreases, then their domain of existence coincides with (8), (9). The components synchronously increasing at $t = 0$ exist if the medium parameters satisfy inequalities (14).

The projection of the density vector J_{chirp} of the spin part of the angular momentum of such waves also depends on the running time:

$$J_{\mathrm{chirp}}(t) = \frac{|\mathcal{E}_-(0)|^2 - |\mathcal{E}_+(0)|^2 + (2\tilde{\mu}^2\tilde{v}^2 k_2\rho_1/\beta)\mathrm{sn}^2(\tilde{v}t,\tilde{\mu})}{16\,\pi k}. \tag{18}$$

In a nongyrotropic medium, the dependence on the running time disappears. The average value of $J_{\mathrm{chirp}}(t)$, as well as for the previous solutions, depends not only on the polarisation of the incident radiation, but also on the parameters of the nonlinear medium:

$$\langle J_{\mathrm{chirp}}(t)\rangle = \frac{|\mathcal{E}_-(0)|^2 - |\mathcal{E}_+(0)|^2 + (2v^2 k_2\rho_1/\beta)[1 - E(\mu)/K(\mu)]}{16\,\pi k}. \tag{19}$$

Particular solutions (2) in the form of degenerate vector breathers [22] propagating along the z axis and satisfying the condition

$$\frac{|\mathcal{E}_+(z,t)|^2}{|\mathcal{E}_-(z,t)|^2} = \frac{\sigma_2 - \rho_1}{\sigma_2 + \rho_1}$$

explicitly depend on the running time and coordinates,

$$|\mathcal{E}_-(z,t)| = B(\sigma_2 + \rho_1)^{1/2}\left|\frac{(1-4a)\mathrm{ch}(zbB^2\beta) + \sqrt{2a}\cos(t\Omega B\sqrt{|\beta/k_2|})}{\sqrt{2a}\cos(t\Omega B\sqrt{|\beta/k_2|}) - \mathrm{ch}(zbB^2\beta)} - \frac{\mathrm{i}b\,\mathrm{sh}(zbB^2\beta)}{\sqrt{2a}\cos(t\Omega B\sqrt{|\beta/k_2|}) - \mathrm{ch}(zbB^2\beta)}\right|, \tag{20}$$

and exist at $\beta k_2 < 0$. Here, $b = \sqrt{8a(1-2a)}$; $\Omega = 2\sqrt{1-2a}$; and the constant $B > 0$ sets the value of the breather's permanent pedestal and its spatiotemporal scale. When $a < 1/2$ and $a > 1/2$, expression (20) describes the Ahmediev breather and the Kuznetsov–Ma soliton, respectively. In the limit $a \to 1/2$, it describes a rational soliton [24, 25]. In (20), B and a are free parameters of the problem.

The projection J_{b} of the density vector of the spin part of the angular momentum of the wave (20) onto the z axis also depends on the running time and coordinate:

$$J_{\mathrm{b}}(z,t) = \frac{\rho_1|\mathcal{E}_-(z,t)|^2}{8\pi k(\sigma_2 + \rho_1)}. \tag{21}$$

For the Akhmediev breather, it is periodic in running time and soliton-like in coordinate, and for the Kuznetsov–Ma soliton, it is soliton-like in running time and periodic in coordinate. In the case of a rational soliton, it behaves as a soliton both in space and in running time. The value of J_{b} averaged over t and z is easy to obtain by taking first the integral over the variable that corresponds to the soliton behaviour of the breather. The values of $J_{\mathrm{b}}(z, t)$ averaged over t and z are the same for all breathers,

$$\langle J_{\mathrm{b}}(z,t)\rangle_{z,t} = \frac{\rho_1 B^2}{8\pi k}, \tag{22}$$

and are nonzero only in a gyrotropic medium at $\rho_1 \neq 0$.

3. Conclusions

A peculiar feature of the obtained expressions for the density vector of the spin part of the angular momentum is the variety of forms of their dependence on the parameters of the nonlinear gyrotropic medium, running time and coordinates. In the absence of nonlinear gyration, the dependences on the last two variables disappear. For all degenerate solutions of both cnoidal waves and breathers, the density vector of the spin part of the angular momentum is proportional to the nonlinear gyration parameter ρ_1 and differs from zero only in an isotropic medium with spatial dispersion of cubic nonlinearity ($\rho_1 \neq 0$).

The results obtained for the density vector of the spin part of the angular momentum for cnoidal waves and vector breathers can contribute to the study of the properties inherent in both the photon and the nonlinear optically active medium, and also be used in integrated optics and optomechanical devices.

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References

1. Toptygin I.N. *Usp. Fiz. Nauk*, **187**, 1007 (2017).
2. Barabanov A.L. *Usp. Fiz. Nauk*, **163**, 75 (1993).
3. Vul'fson K.S. *Usp. Fiz. Nauk*, **152**, 667 (1987).
4. Sokolov I.V. *Usp. Fiz. Nauk*, **161**, 175 (1991).
5. Stewart A.M. *Eur. J. Phys.*, **26**, 635 (2005).
6. Barnett S.M. *J. Mod. Opt.*, **57**, 1339 (2010).
7. Barnett S.M. et al. *J. Opt.*, **18**, 064004 (2016).

8. Mandel L., Wolf E. *Optical Coherence and Quantum Optics* (Cambridger: Cambridge University Press, 1995; Moscow: Fizmatlit, 2000).
9. Bialynicki-Birula I., Bialynicka-Birula Z. *J. Opt*., **13**, 064014 (2011).
10. Akhiezer A.I., Berestetsky V.B. *Quantum Electrodynamics* (New York: Consultants Bureau, Inc., 1957; Moscow: Nauka, 1981).
11. Crichton J.H., Marston P.L. http://ejde.math.unt.edu.
12. Bliokh K.Y., Nori F. *Phys. Reports*, **592**, 1 (2015).
13. Loke V.L.Y., Asavei T., Parkin S., Heckenberg N.R., Rubinsztein-Dunlop H., Nieminen T.A., in *Twisted Photons, Applications of Light with Orbital Angular Momentum*. Ed. by J.P. Torres, L. Torner (Berlin: Wiley-VCH, 2011) Ch. 6, pp 93–116.
14. Beijersbergen M.W., Coerwinkel R.P.C., Kristesen M., Woerdman J.P. *Opt. Commun*., **112**, 321 (1994).
15. Bazhenov V.Yu., Vasnetsov M.V., Soskin M.S. *JETP Lett*., **52**, 429 (1990) [*Pis'ma Zh. Eksp. Teor. Fiz*., **52**, 1037 (1990)].
16. Courtial J., Dholakia K., Allen L., Padgett M.J. *Opt. Commun.*, **144**, 210 (1997).
17. Devlin R.C., Ambrosio A., Rubin N.A., Balthasar Mueller J.P., Capasso F. *Science*, **358** (6365), 896 (2017).
18. He L., Li H., Li M. *Sci. Adv*., **2**, e1600485 (2016).
19. Makarov V.A., Perezhogin I.A., Petnikova V.M., Potravkin N.N., Shuvalov V.V. *Quantum Electron*., **42**, 117 (2012) [*Kvantovaya Elektron*., **42**, 117 (2012)].
20. Makarov V.A., Petnikova V.M., Potravkin N.N., Shuvalov V.V. *Phys. Wave Phenom*., **21**, 264 (2013).
21. Makarov V.A., Petnikova V.M., Potravkin N.N., Shuvalov V.V. *Quantum Electron*., **42**, 1118 (2012) [*Kvantovaya Elektron*., **42**, 1118 (2012)].
22. Makarov V.A., Petnikova V.M., Ryzhikov P.S., Shuvalov V.V., Yadvichuk A.V. *Phys. Wave Phenom.*, **25**, 20 (2017).
23. Gradshteyn I.S., Ryzhik I.M. *Tables of Integrals, Series and Products* (San Diego, CA: Academic Press, 2000; Moscow: Nauka, 1971).
24. Kibler B., Fatome J., Finot C., Millot G., Dias F., Genty G., Akhmediev N., Dudley J.M. *Nature Phys*., **6**, 790 (2010).
25. Kibler B., Fatome J., Finot C., Millot G., Genty G., Wetzel B., Akhmediev N., Dias F., Dudley J.M. *Sci. Reports*, **2**, 463 (2012).