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Adaptive correction of the image of an incoherent source object

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Abstract. **The paper examines the operation of an optical system forming an image of an incoherent source object under conditions of atmospheric noise. The effect of atmospheric turbulence is considered. The calculations are performed in the Fresnel–Huygens approximation. The characteristics of the image for 'long' and 'short' exposures are investigated. The issues related to the adaptive correction of the general light wavefront slopes are discussed. The limits of applicability of phase adaptive correction using a reference source are shown.**

Keywords: *turbulence, image, adaptive correction, reference source, strong fluctuations.*

1. Introduction

One of the tasks of optics is to observe objects with extremely high angular and spatial resolutions, determined only by the radiation wavelength λ and the telescope aperture diameter D. The most important distorting factor is atmospheric turbulence. This factor, which determines in the optical range the so-called correlation radius of phase atmospheric distortions r_0, or Fried radius, can vary over a fairly wide range. Therefore, in ordinary observations, practically any telescope will produce an angular resolution, specified by the parameter λ/r_0 and independent of the size of the telescope, while the diffraction resolution of the telescope, equal to λ/D, is usually much higher. At present, the problem of increasing the angular resolution during image formation is solved by several methods, including those based on wavefront correction, i.e., adaptive phase [1] and digital post-detector [2–4] methods (for example, aperture synthesis methods) as well as methods based on purely engineering solutions.

Usually, the problem of improving the quality of image formation is solved under the following initial conditions: the object of observation lies in a small angle, 5″–10″, i.e. the condition of isoplanatism is met [1]; the recording time is shorter than the time of the atmosphere 'frozenness' (the condition of statistical independence of atmospheric distortions); recording is carried out in a narrow spectral range $\Delta\lambda$ near the selected wavelength: $\Delta\lambda = \lambda r_0/D$ (quasi-monochromaticity condition). In a real situation, the object is illuminated by sunlight; a variant of incoherent optical illumination of the object is possible, in which no speckles appear in the field formed on the object.

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2. Formulation of the calculation problem

We set the task to write the expression for the distribution of the average image intensity of an incoherent source object observed through a layer of a turbulent medium. Consider the following scheme. Let an incoherent radiating object or an object illuminated by sunlight be located inside a turbulent atmosphere at a distance L from the receiving telescope (lens). The telescope has an aperture with a diameter D and a focal length F. The image formation will be described in the approximation of the Fresnel–Huygens method [5, 6]. Then the distribution of the instantaneous power density of the radiation in the sharp image plane $X_{\rm img}$ can be expressed as

$$I_{\rm img}(\boldsymbol{r}) = \iint_\Sigma {\rm d}^4\rho_{1,2}\, W(\rho_1/R_{\rm a})\, W^*(\rho_2/R_{\rm a})$$

$$\times G_0(0,\boldsymbol{\rho}_1; -X_{\rm img},\boldsymbol{r})\, G_0^*(0,\boldsymbol{\rho}_2; -X_{\rm img},\boldsymbol{r}) \exp\Big(-{\rm i}k\boldsymbol{r}\frac{\boldsymbol{\rho}_1-\boldsymbol{\rho}_2}{F}\Big)$$

$$\times \iint {\rm d}^4 r_{1,2}\, U_{\rm ob}(\boldsymbol{r}_1)\, U^*_{\rm ob}(\boldsymbol{r}_2)\, G(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1)\, G^*(L,\boldsymbol{r}_2;0,\boldsymbol{\rho}_2), \quad (1)$$

where $W(\rho_1/R_{\rm a})$ is the aperture function of the telescope; Σ is the receiving aperture; $G_0(0,\boldsymbol{\rho}_1; -X_{\rm ing},\boldsymbol{r})$ is Green's function of free space (inside the telescope); $G(L,\boldsymbol{r}_1; 0,\boldsymbol{\rho}_1)$ is Green's function of the turbulent medium in the space between the object and the telescope; k is the wave number of radiation; $\boldsymbol{\rho}_{1,2}$ and $\boldsymbol{r}_{1,2}$ are two-dimensional vectors; and $R_{\rm a}$ is the radius of the telescope aperture.

First of all, in expression (1), we perform averaging over the fluctuations of the radiation source, and assume that the radiation field $U_{\rm ob}(\boldsymbol{r}_1)$ is incoherent; therefore, for the coherence function we use the approximation:

$$\langle U_{\rm ob}(\boldsymbol{r}_1)U^*_{\rm ob}(\boldsymbol{r}_2)\rangle = I_{\rm ob}(\boldsymbol{r}_1)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2), \quad (2)$$

where $I_{\rm ob}(\boldsymbol{r}_1)$ is the surface brightness of the source; and $\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)$ is the Dirac delta function.

As a result, expression (1) is rewritten as

$$I_{\rm img}(\boldsymbol{r}) = \iint_\Sigma {\rm d}^4\rho_{1,2}\, W(\rho_1/R_{\rm a})\, W^*(\rho_2/R_{\rm a})$$

$$\times G_0(0,\boldsymbol{\rho}_1; -X_{\rm img},\boldsymbol{r})\, G_0^*(0,\boldsymbol{\rho}_2; -X_{\rm img},\boldsymbol{r}) \exp\Big(-{\rm i}k\boldsymbol{r}\frac{\boldsymbol{\rho}_1-\boldsymbol{\rho}_2}{F}\Big)\times$$

$$\times\iint d^2r_1 I_{ob}(\boldsymbol{r}_1)\,G(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1)\,G^*(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2), \quad (3)$$

which somewhat simplifies calculations.

Expression (3) corresponds to the instantaneous value of the power density distribution in the image of an incoherent source having a brightness function of form (2). The, using the phase approximation [7] of the Fresnel–Huygens method generalised for randomly inhomogeneous media, we have

$$G(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1)\,G^*(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2) = G_0(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1)\,G_0^*(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)$$
$$\times\exp\{i[S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)]\}, \quad (4)$$

where $S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1)$ and $S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)$ are phase fluctuations of an elementary spherical wave propagating from a point with coordinates $L,\boldsymbol{r}_1$ to a point with coordinates $0,\boldsymbol{\rho}_1$; and

$$G_0(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) = \frac{\exp[ikL + ik(\boldsymbol{\rho}_1 - \boldsymbol{r}_1)^2/(2L)]}{L}.$$

Analysis of expressions (1), (3) and (4) shows that phase adaptive correction of turbulent distortions requires the use of a signal from the reference source. The reference source of a spherical wave located in the plane of the observed source object within the range of isoplanatism of phase fluctuations is optimal [8–10]. Below, we will use such a point reference source located on the optical axis at point $(L, 0)$. As a result, we obtain that the correction signal generated from the reference source is the phase of a spherical wave of form $S(L,0;0,\boldsymbol{\rho}_1)$. In this case, the size of the most incoherent object should be less than the size of the isoplanatism region [10, 11] cause by the turbulent medium.

If the object is located far enough from the turbulent layer, then at the entrance to it the wavefront of the optical wave from the object is almost flat. This situation takes place in astronomy and in the observation of satellites. In this case, expression (1) can be rewritten as

$$I_{img}(\boldsymbol{r}) = \iint_\Sigma d^4\rho_{1,2}\,W(\rho_1/R_a)\,W^*(\rho_2/R_a)$$
$$\times G_0(0,\boldsymbol{\rho}_1;-X_{img},\boldsymbol{r})\,G_0^*(0,\boldsymbol{\rho}_2;-X_{img},\boldsymbol{r})\exp\Big(-ik\boldsymbol{r}\frac{\boldsymbol{\rho}_1-\boldsymbol{\rho}_2}{F}\Big)$$
$$\times\exp\{i[S(0,\boldsymbol{\rho}_1) - S(0,\boldsymbol{\rho}_2)]\}, \quad (5)$$

where $S(0,\boldsymbol{\rho}_1)$ is the phase fluctuations in a plane wave caused by the action of atmospheric turbulence. In this case, the optimal reference wave [1] is a plane wave incident on the aperture within an angle not exceeding the angular size of the isoplanatic region.

In the following, we will consider only the case when the incoherent source object is located inside a turbulent medium. The object itself is illuminated by sunlight, and to produce a reference source, one can use additional lighting. For example, radiation back-reflected [1, 11] by an angular reflector can be used as a reference wave. The size d of the corner reflector must be chosen from the condition $d < \sqrt{\lambda L}$ (here *1* is the radiation wavelength of the backlight), i.e. it should not exceed the size of the first Fresnel zone.

As a result of substitution of expression (4) into (3) we obtain

$$I_{img}(\boldsymbol{r}) = \iint_\Sigma d^4\rho_{1,2}\,W(\rho_1/R_a)\,W^*(\rho_2/R_a)$$
$$\times G_0(0,\boldsymbol{\rho}_1;-X_{img},\boldsymbol{r})\,G_0^*(0,\boldsymbol{\rho}_2;-X_{img},\boldsymbol{r})\exp\Big(-ik\boldsymbol{r}\frac{\boldsymbol{\rho}_1-\boldsymbol{\rho}_2}{F}\Big)$$
$$\times\iint d^2r_1 I_{ob}(\boldsymbol{r}_1)\,G_0(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1)\,G_0^*(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)$$
$$\times\exp\{i[S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)]\}. \quad (6)$$

To pass from an instantaneous value to an average value of the received intensity of an object image, we perform averaging in (6), assuming that the phase fluctuations $S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1)$ are a Gaussian random field. Then, as a result of averaging over random turbulent fluctuations, expression (6) takes the form (angle brackets denote averaging)

$$\langle I_{img}(\boldsymbol{r})\rangle = \frac{1}{L^2X_{img}^2}\iint_\Sigma d^4\rho_{1,2}\,W(\rho_1/R_a)\,W^*(\rho_2/R_a)$$
$$\times\exp\Big\{ik\Big[\frac{(\boldsymbol{\rho}_1-\boldsymbol{r})^2}{2X_{img}} - \frac{(\boldsymbol{\rho}_2-\boldsymbol{r})^2}{2X_{img}}\Big]\Big\}\exp\Big(-ik\boldsymbol{r}\frac{\boldsymbol{\rho}_1-\boldsymbol{\rho}_2}{F}\Big)$$
$$\times\iint d^2r_1 I_{ob}(\boldsymbol{r}_1)\exp\Big\{ik\Big[\frac{(\boldsymbol{\rho}_1-\boldsymbol{r}_1)^2}{2L} - \frac{(\boldsymbol{\rho}_2-\boldsymbol{r}_1)^2}{2L}\Big]\Big\}$$
$$\times\langle\exp\{i[S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)]\}\rangle. \quad (7)$$

It is easy to show that the fluctuation phase term $\langle\exp\{i[S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)]\}\rangle$, calculated for 'long' exposure [12], is expressed in terms of the structural phase function as:

$$\langle\exp\{i[S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)]\}\rangle$$
$$= \exp\Big\{-\frac{1}{2}\langle[S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)]^2\rangle\Big\}. \quad (8)$$

In addition, for the Kolmogorov turbulence model [5], the structural function of the phase $D_S(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)$ is written in the form

$$\langle[S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)]^2\rangle$$
$$= D_S(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) = 6.88\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^{5/3}}{r_0^{5/3}}, \quad (9)$$

where r_0 is the coherence radius (Fried parameter) for a spherical wave, which is calculated [5, 6] using the formula

$$r_0 \approx \Big[0.42k^2\int_0^L C_n^2(\xi)(1-\xi/L)^{5/3}d\xi\Big]^{-3/5}.$$

If in (7) we make the replacement of variables ($\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 = \boldsymbol{\rho}$, $\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2 = 2\boldsymbol{R}$), and also use the thin lens formula ($F^{-1} = L^{-1} + X_{img}^{-1}$), then the distribution of the average image intensity under the conditions of long exposure [12] will have the form

$$\langle I(X_{img},\boldsymbol{r})\rangle = \frac{1}{L^2X_{img}^2}\iint_\Sigma d^2\rho\iint_\Sigma d^2R\,W\Big(\frac{\boldsymbol{R}+\boldsymbol{\rho}/2}{R_a}\Big)$$
$$\times W^*\Big(\frac{\boldsymbol{R}-\boldsymbol{\rho}/2}{R_a}\Big)\exp\Big[-ik\boldsymbol{r}\frac{\boldsymbol{\rho}}{X_{img}} - 3.44\Big(\frac{\rho}{r_0}\Big)^{5/3}\Big]\times$$

$$\times \iint \mathrm{d}^2 r_1 I(\boldsymbol{r}_1) \exp\left(-\mathrm{i}k\boldsymbol{r}_1 \frac{\boldsymbol{\rho}}{L}\right). \tag{10}$$

Next, we introduce the overlap function [12] of the receiving aperture, $K_\mathrm{a}(\boldsymbol{\rho})$, which characterises the first internal integral in (10), in the form

$$K_\mathrm{a}(\boldsymbol{\rho}) = \iint \mathrm{d}^2 R W\left(\frac{\boldsymbol{R}+\boldsymbol{\rho}/2}{R_\mathrm{a}}\right) W\left(\frac{\boldsymbol{R}-\boldsymbol{\rho}/2}{R_\mathrm{a}}\right). \tag{11}$$

As a result, from (10) we obtain

$$\langle I(X_\mathrm{img}, \boldsymbol{r})\rangle = \frac{1}{L^2 X_\mathrm{img}^2} \iint_\Sigma \mathrm{d}^2 \rho K_\mathrm{a}\left(\frac{\boldsymbol{\rho}}{R_\mathrm{a}}\right)$$
$$\times \exp\left[-\mathrm{i}k\boldsymbol{r}\frac{\boldsymbol{\rho}}{F} - 3.44\left(\frac{\rho}{r_0}\right)^{5/3}\right]$$
$$\times \iint \mathrm{d}^2 r_1 I_\mathrm{ob}(\boldsymbol{r}_1) \exp\left(-\mathrm{i}k\boldsymbol{r}_1 \frac{\boldsymbol{\rho}}{L}\right). \tag{12}$$

3. Spatial spectrum of the image

If now in the final expression (12) we make a transition to the corresponding spatial Fourier spectrum

$$J(X_\mathrm{img}, \boldsymbol{f}) = \iint \mathrm{d}^2 r \, \langle I_\mathrm{ob}(X_\mathrm{img}, \boldsymbol{r})\rangle \exp(\mathrm{i}2\pi \boldsymbol{f}\boldsymbol{r}), \tag{13}$$

then in the approximation of long exposure [12], according to Fried's definition, the spatial spectrum of the image is obtained in the form:

$$J(X_\mathrm{img}, \boldsymbol{f}) = \frac{1}{X_\mathrm{img}^2 L^2} K_\mathrm{a}\left(\frac{\lambda X_\mathrm{img}\boldsymbol{f}}{R_\mathrm{a}}\right) \iint \mathrm{d}^2 r_1 I_\mathrm{img}(\boldsymbol{r}_1)$$
$$\times \exp\left(-\mathrm{i}k\boldsymbol{r}_1 \frac{\lambda X_\mathrm{img}\boldsymbol{f}}{L}\right) \exp\left[-3.44\left(\frac{\lambda X_\mathrm{img}\boldsymbol{f}}{r_0}\right)^{5/3}\right]. \tag{14}$$

Here f is the spatial frequency modulus and the condition

$$\iint \mathrm{d}^2 r \exp(\mathrm{i}k\boldsymbol{r}\boldsymbol{\rho}/X_\mathrm{img}) \exp(\mathrm{i}2\pi \boldsymbol{f}\boldsymbol{r}) = \delta(\boldsymbol{\rho} - \lambda X_\mathrm{img}\boldsymbol{f}) \tag{15}$$

is used.

Next, we introduce the function

$$\chi\left(\frac{\lambda X_\mathrm{img} f}{r_0}\right) = \exp\left[-3.44\left(\frac{\lambda X_\mathrm{img} f}{r_0}\right)^{5/3}\right], \tag{16}$$

which is an optical transfer function (OTF) caused by the action of turbulence.

Thus, the spatial Fourier transform of the distribution of the average intensity (13) for an incoherent source object is the product [13] of three factors:

$$J(X_\mathrm{img}, \boldsymbol{f}) = \frac{1}{X_\mathrm{img}^2 L^2} K_\mathrm{a}\left(\frac{\lambda X_\mathrm{img}\boldsymbol{f}}{R_\mathrm{a}}\right)$$
$$\times \chi\left(\frac{\lambda X_\mathrm{img} f}{r_0}\right) J_\mathrm{ob}\left(\boldsymbol{f}\frac{X_\mathrm{img}}{L}\right). \tag{17}$$

Here,

$$J_\mathrm{ob}\left(\boldsymbol{f}\frac{X_\mathrm{img}}{L}\right) = \iint \mathrm{d}^2 r_1 I_\mathrm{ob}(\boldsymbol{r}_1) \exp\left(-\mathrm{i}2\pi \boldsymbol{r}\boldsymbol{f}\frac{X_\mathrm{img}}{L}\right)$$

is the spatial (angular) emission spectrum of an incoherent source.

It should be noted that only for an incoherent source (object) one can one pass from the expression for the distribution of average intensity (10) to the spatial spectrum in such a way that the spatial spectrum of the image (17) will be the product of the following three factors: the angular emission spectrum of the object with the scaling of the spatial frequency $J_\mathrm{ob}(X_\mathrm{img}\boldsymbol{f}/L)$, Fourier transform of the point spread function (PSF) ψ for the atmosphere, i.e., OTF of the form $\exp[-3.44(\lambda X_\mathrm{img} f)/r_0)^{5/3}]$, and the spatial transmittance spectrum of the aperture telescope, $K_\mathrm{a}(\lambda X_\mathrm{img}\boldsymbol{f}/R_\mathrm{a})$, which is the Fourier transform of $K_\mathrm{a}(\boldsymbol{\rho})$ (11).

For a distance $L \gg F$, expression (17) can be simplified:

$$J(F, \boldsymbol{f}) = \frac{1}{F^2 L^2} K_\mathrm{a}\left(\frac{\lambda F\boldsymbol{f}}{R_\mathrm{a}}\right) \chi\left(\frac{\lambda F f}{r_0}\right) J_\mathrm{ob}\left(\boldsymbol{f}\frac{F}{L}\right). \tag{18}$$

For the expression for the spatial spectrum to look like (17) and (18), it is necessary that the PSF of the turbulent atmosphere [12] depend on the coordinates of the points $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ within the aperture Σ only through their difference, i.e. be of form $\exp[-3.44(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|/r_0)^{5/3}]$. For long exposure, this condition is met, but only if the atmospheric turbulence is described by the isotropic Kolmogorov–Obukhov model.

Moreover, in describing the spectrum of an image of an incoherent source in the form of (18), it is possible to improve the quality of the formed image by an inverse correction filter [3, 4]. In this case, such a correction filter [14] can be calculated by the formula

$$F_\mathrm{corr}(f) = F^{-1}(\psi) = \exp\left[3.44\left(\frac{\lambda X_\mathrm{img} f}{r_0}\right)^{5/3}\right]. \tag{19}$$

Then, using a high-speed video camera and a computer, one can implement one of the methods for improving image quality, for example, the inverse filtering method [14, 15] for the image spectrum.

4. Comparison of cases of short and long exposures

It is known that the operation of transition from the instantaneous value of the intensity distribution of an image of form (6) to its average value can be performed with other observation regimes, including, according to Fried's classification [12], with the so-called short exposure. In this case, as a rule, the PSF or its Fourier transform – the spatial spectrum (frequency-contrast characteristic) – is written in the Fried approximation. It should be noted that the expression for the PSF under short exposure was obtained by Fried [12] with well-defined approximations. By itself, Fried's short exposure is, in fact, the result of calculating the average image intensity during the correction of random slopes of the wavefront [1]. If we use Noll's notation [13], then the expression for the residual phase distortions after correction of the wavefront slopes will be written as

$$\Delta S(0, \boldsymbol{\rho}_1) = S(L, \boldsymbol{r}_1; 0, \boldsymbol{\rho}_1) - 2a_2\frac{x_1}{R_\mathrm{a}} - 2a_3\frac{y_1}{R_\mathrm{a}}, \tag{20}$$

where a_2 and a_3 are the expansion coefficients (slopes along the x and y axes) of phase fluctuations in the Zernike polynomials.

In [1] we showed that as a result of this correction, the structural function of residual phase fluctuations can be written, using only the first eight Zernike polynomials in the expansion of phase fluctuations, in the form:

$$D_{\Delta S}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \left\langle \left\{ [S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_1) - S(L,\boldsymbol{r}_1;0,\boldsymbol{\rho}_2)] - 2\left(a_2\frac{x_1-x_2}{R_a} + a_3\frac{y_1-y_2}{R_a}\right)\right\}\right\rangle = 6.88\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^{5/3}}{r_0^{5/3}}$$
$$-\frac{4\langle a_2^2\rangle}{R_a^2}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2 - 2\langle a_2 a_8\rangle$$
$$\times\left\{\frac{12\sqrt{2}}{R^4}\left[\rho_1^4+\rho_2^4-(\rho_1^2+\rho_2^2)\boldsymbol{\rho}_1\boldsymbol{\rho}_2 - \frac{8\sqrt{2}}{R^2}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2\right]\right\}, \quad (21)$$

where the notations from [13] are used.

Expression (21) was written for the case when $\langle a_2^2\rangle = \langle a_3^2\rangle$ and $\langle a_2a_8\rangle = \langle a_3a_7\rangle$, and this corresponds to the condition of isotropy of the turbulence spectrum [1, 5]. For further analysis, we use the results of calculations [13, 16] for the dispersion $\langle a_2^2\rangle$ and correlation $\langle a_2a_8\rangle$:

$$\langle a_2^2\rangle = 1.42(R/r_0)^{5/3}, \quad \langle a_2a_8\rangle = -0.045(R/r_0)^{5/3}. \quad (22)$$

First of all, it is interesting to compare expression (21) with other results. Thus, in papers [12, 15, 16], the OTF for an optical system operating through a layer of a turbulent atmosphere was analysed in the absence of correlation between corrected wavefront slopes and higher phase fluctuations. In the notations used in this paper, this corresponds to the fact that $\langle a_2a_8\rangle \equiv 0$ and $\langle a_3a_7\rangle \equiv 0$. Under these assumptions, Fried's formula for the structural function of the residual phase (20) with short exposure [12] is expressed as:

$$D_{\Delta S}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = 6.88\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^{5/3}}{r_0^{5/3}} - 5.68\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2}{R^{1/3}r_0^{5/3}}. \quad (23)$$

This expression is slightly different from the widely used expression [17]

$$D_{\Delta S}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2) = 6.88\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^{5/3}}{r_0^{5/3}}\left[1-\left(\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|}{R_a}\right)^{1/3}\right]. \quad (24)$$

However, the reduced Fried expression (23) gives correct results only for $R_a \leqslant 2r_0$.

It should be noted that often expressions for the structural function of the residual phase with short exposure are written in the form

$$D_{\Delta S}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2) = 6.88\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^{5/3}}{r_0^{5/3}}\left[1-\alpha\left(\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|}{R_a}\right)^{1/3}\right],$$

where the parameter α takes values in the range of 0.5–1.0.

If we calculate the $D_{\Delta S}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ more precisely, for example, taking into account all the terms in (21), including correlations of type $\langle a_2a_8\rangle$, then we obtain

$$D_{\Delta S}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = 6.88\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^{5/3}}{r_0^{5/3}} - 6.70\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2}{R_a^{1/3}r_0^{5/3}} + 0.38\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^4}{R_a^{7/3}r_0^{5/3}} + 4.56\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2R^2}{R_a^{7/3}r_0^{5/3}}. \quad (25)$$

It is easy to see that according to expression (25), the structural function of the residual phase $D_{\Delta S}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ depends not only on $|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|$, but also on $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$, i.e., anisotropy of the properties of this function takes place. This means that the transition from the PSF to the spatial spectrum will no longer give such a simple result as formula (18).

Let us analyse the consequences of using formula (24) instead of formulas (23) and (25). We rewrite the expression for the structural function of the residual phase in the simplest form:

$$D_S(\boldsymbol{\rho}) = 6.88\frac{\rho^{5/3}}{r_0^{5/3}} \quad (26)$$

for long exposure, and in the form

$$D_{\Delta S}(\boldsymbol{\rho}) = 6.88\frac{\rho^{5/3}}{r_0^{5/3}}\left[1-0.826\left(\frac{\rho}{R_a}\right)^{1/3}\right] \quad (27)$$

for a short exposure according to Fried's model. At the same time, instead of formula (27), formula (24) is usually used.

We now write the resulting formula (25) in the form

$$D_{\Delta S}(\boldsymbol{\rho},\boldsymbol{R}) = 6.88\frac{\rho^{5/3}}{r_0^{5/3}}$$
$$\times\left[1-0.97\left(\frac{\rho}{R_a}\right)^{1/3}+0.055\left(\frac{\rho}{R_a}\right)^{7/3}+0.66\left(\frac{\rho}{R_a}\right)^{1/3}\left(\frac{R}{R_a}\right)^2\right]. \quad (28)$$

Let's compare expressions (24), (26)--(28), by calculating the ratio of the structural functions of the residual phase for short exposure (24), (27), (28) to the structural function of the phase for long exposure (26); finally, we obtain a set of functions

$$\varphi(x) = \frac{D_{\Delta S}^{\text{short}}(\boldsymbol{\rho})}{D_S^{\text{long}}(\boldsymbol{\rho})}, \quad (29)$$

where the variable $x = \rho/R_a$ changes in the region of its determination from 0 to 2. In these notations, according to the (correct) Fried calculations, from (27) and (26), we have

$$\varphi_1(x) = 1 - 0.826x^{1/3}, \quad (30)$$

and using the widely used scheme (24) with regularisation [14, 15], we obtain

$$\varphi_2(x) = 1 - x^{1/3},$$
$$\varphi_3(x) = 1 - 0.5x^{1/3}. \quad (31)$$

Finally, our refined formula (28) gives

$$\varphi_4(x) = 1 - 0.974x^{1/3} + 0.055x^{7/3} + 0.33x^{1/3}(R/R_a)^2. \quad (32)$$

It should be clarified that the formulae for the functions $\varphi_2(x)$ and $\varphi_3(x)$) differ only by a factor. This is due to the fact that for large distances in a turbulent atmosphere, amplitude fluctuations appear along with phase fluctuations. It is considered [18] that in this case it is more correct to use formula (31) for $\varphi_3(x)$.

The results of calculations by formulae (30)–(32) are presented in Fig. 1, with the ratio D/r_0 being the parameter determining the 'force of turbulence'. Comparison of the curve shown in Fig. 1 gives the opportunity to draw some conclusions, namely:

– according to formula (31), the curve $\varphi_2(x)$ in the region $x > 1$ lies in the negative region; this is, generally speaking, non-physical, since it means that the effect of atmospheric turbulence on the path leads to distortions that increase the field amplitude to values greater than that when the field propagates in a vacuum;

– according to formula (31), compared with the curve $\varphi_2(x)$ the curve $\varphi_3(x)$ preserves a lot of distortion as a result of filtering and thus provides a low correction efficiency;

– according to formula (32), the curve $\varphi_4(x)$ in the entire range of x values provides a high correction efficiency and remains physical, without going into the negative region; and

– according to the 'correct' Fried formula (30), the curve $\varphi_1(x)$ lies in the negative (non-physical) region only at the very end of the interval of x values and describes the correction much more correctly compared to the $\varphi_2(x)$ curve.

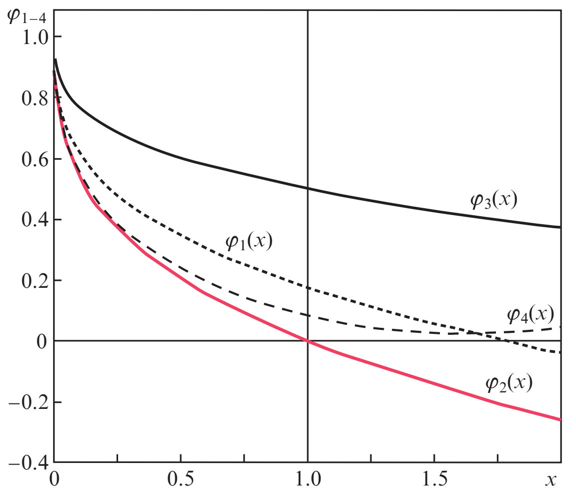

Figure 1. Ratio of the structural functions of the residual phase during short exposure to the structural function of the phase during long exposure.

In many ways, similar results were obtained by Vitsekhovich [19], but he used an approach to similar calculations without a transition to the phase description. In this work, another alternative approach was implemented to calculate the intensity distribution of the image during the correction the wavefront slopes. Random slopes are corrected using the measurement data on the position of the centre of gravity of the image within the aperture of a certain radius $\mathcal{R}$, which is consistent with the level of turbulence. It was shown that the structural function of residual phase distortions (25) now has the form

$$D_{\Delta S}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = 6.88\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^{5/3}}{r_0^{5/3}} + 5.32\frac{(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2}{\mathcal{R}^{1/3}r_0^{5/3}}$$

$$-11.4\mathcal{R}^{-1/3}r_0^{-5/3}\left\{\frac{\rho_1^2}{(\rho_1/\mathcal{R}+1)^{1/3}} - \boldsymbol{\rho}_1\boldsymbol{\rho}_2\right.$$

$$\left.\times[(\rho_1/\mathcal{R}+1)^{-1/3} + (\rho_2/\mathcal{R}+1)^{-1/3}] + \frac{\rho_2^2}{(\rho_2/\mathcal{R}+1)^{1/3}}\right\}, \quad (33)$$

where $\mathcal{R} \leqslant R_a$.

If we compare this expression with (28), then, as shown in [1], we arrive at almost the same results from the point of view of correction. In this case, the optimal size of the aperture $\mathcal{R}$ depends on the ratio of R_a and r_0. So, for $R_a < 6r_0$ the optimal size is $\mathcal{R} \approx r_0/2$.

Thus, to ensure optimal correction, the aperture must correspond to the propagation conditions on the path and 'follow' the coherent part of the phase front.

5. Change of coherence radius during correction

The process of image formation at different exposures and in using adaptive phase correction can be viewed in terms of increasing the coherence radius. In the case of the formation of an incoherent image, one should of course speak of an increase in the coherence of the receiving atmosphere–telescope path. We investigate this feature of applying phase correction as the possibility of increasing the coherence region as a result of even partial phase correction. Moreover, speaking of the size of the isoplanatism zone of a system that builds an image, it is certainly more accurate to calculate it using the Fried radius value obtained by correcting the wavefront slopes, i.e., for short exposure [12]. This is primarily due to the fact that the total wavefront slope does not affect the quality of the instant image, but only leads to its displacement. In addition, the value of the total wavefront slope for the system is significantly larger than the correlation radii for higher wavefront aberrations [1, 13, 16–18]. The size of the isoplanatism zone for the wavefront slope (sometimes this area is also called the isokinetic zone) depends not only on the parameters of turbulence, but also on the size of the receiving aperture (telescope aperture) [16–18].

Thus, if we use short exposure [12], then for the resulting field we obtain the actual change in the radius of radiation coherence. Let us estimate the effect of the correction of the wavefront slopes using, for example, formulae (26) and (27) for the structural function of the phase, which yield the relation

$$6.88(\rho/\tilde{r}_0)^{5/2} = 6.88(\rho/r_0)^{5/2}[1 - 0.826(\rho/R_a)^{1/3}] \quad (34)$$

(here $\tilde{r}_0$ is the coherence radius of the field after correction of the slopes), which leads to the formula

$$\tilde{r}_0 = r_0[1 - 0.826(\rho/R_a)^{1/3}]^{-3/5}. \quad (35)$$

Formula (35) shows that an increase in the coherence radius depends on the observation point, namely:

– on the axis of the system (at $\rho = 0$), the coherence radius after correction of the slopes coincides with the coherence radius without correction ($\tilde{r}_0 = r_0$), i.e. there is no increase;

– at $\rho = R_a$, we obtain $\tilde{r}_0 = 2.8r_0$; and

– at $\rho = 2R_a$, the value of $\tilde{r}_0$ is not defined.

From the above refined calculation by formula (28) for the structural function of the residual phase in the case of short exposure, according to formulae (32) and (26), we have

$$\tilde{r}_0 = r_0[1 - 0.97(\rho/R_a)^{1/3} + 0.55(\rho/R_a)^{7/3}$$

$$+0.33(\rho/R_a)^{1/3}(\mathcal{R}/R_a)^2]^{-3/5}. \quad (36)$$

Then, by analogy with the previous one, we obtain:

– on the axis of the system (at $\rho = 0$), the coherence radius does not change, $\tilde{r}_0 = r_0$, i.e. there is no increase;

– at $\rho = R_a$, we have $\tilde{r}_0 = 1.7r_0$; and
– at $\rho = 2R_a$, we have $\tilde{r}_0 = 1.92r_0$.

Such an increase in the coherence radius actually takes place, since the correction of the slopes has the greatest effect on the periphery. We obtain that on average the use of short exposure, or, equivalently, the use of wavefront slope correction actually increases the effective size of the coherent zone by about 1.8 times.

If we use an adaptive system that performs a partial phase correction of, say, several Zernike modes, then we can similarly estimate the increase in the coherence radius using Noll's formulae [13].

6. Range of applicability of phase correction

Of course, like any other method, phase adaptive correction requires determining the range of its applicability from the point of view of the parameters describing turbulence. It has already been shown [20, 21] that for horizontal paths, the adaptive phase correction becomes ineffective with increasing fluctuations. This occurs when the coherence radius due to turbulence becomes smaller than the size of the first Fresnel zone, which corresponds to the condition

$$kr_0^2/L < 1. \tag{37}$$

As a rule, it is on horizontal atmospheric paths that one has to deal with small Fried radii and the manifestation of strong phase fluctuations. The level of turbulent distortions on the atmospheric path is characterised by the Fried radius r_0 or the ratio D/r_0. This parameter can be measured, for example, directly in the experiment with the help of a wavefront sensor operating according to the differential technique [22].

It should be noted that there is no generally accepted concept of 'strong' fluctuations. Let us try to introduce it. Using the results of the theory of image formation under turbulent distortion, the maximum achievable angular resolution is estimated as

$$\Theta \approx \lambda(D^{-2} + r_0^{-2})^{1/2}. \tag{38}$$

It follows from this formula that when $D < r_0$, the resolution corresponds to the diffraction resolution, but if the situation is reversed, then $\Theta \approx \lambda/r_0$. In reality, when performing a correction, the most that can be done is to obtain $r_0 = D$.

Below, we can use the following classification of levels of turbulence:
– at $D/r_0 < 4$, the distortions are 'weak';
– at $4 < D/r_0 < 10$, the distortions are 'moderate';
– at $D/r_0 > 10$, the distortions are 'strong'; and
– at $D/r_0 > 15-20$, the phase fluctuations are ''very strong'.

The question arises of how large D/r_0 should be to trigger a strong turbulent broadening. Naturally, if $D < r_0$, then the turbulent broadening is weak. It is also necessary to clarify why we use $D/r_0 < 4$ to determine the boundary of 'weak' fluctuations. At this level of fluctuations, all phase fluctuations are reduced only to the wavefront slopes; therefore, only the image position is shifted, which can be easily eliminated by correcting only the wavefront slopes (these are Zernike polynomials with numbers 2 and 3). In the range $D/r_0 = 4-10$, correction of only the lowest modes is needed (these are slopes, defocusing and coma). Calculations show that at $D = 4r_0$ the best result is reached by correcting the slopes (lower modes); the same effect should be expected from the post-detector correction. If $4 < D/r_0 < 10$, then the correction no longer leads to full recovery, and the resolution of the system increases approximately by two to three times.

In addition, as is well known, any adaptive optical system is a dynamic system with its own operation frequency. Also, according to theory, the required frequency of the adaptive system for carrying out a full phase correction is determined by the formulae [23, 24], similar to the following:

$$f_G \approx 0.43\frac{v_\perp}{r_0}, \tag{39}$$

where $v_\perp$ is the transverse wind speed.

At the same time, if any adaptive correction phase system is implemented, a positive result can be achieved only when certain conditions are met in the atmosphere. One of them is the following condition: on the path, the angle of isoplanatism must be greater than the angular resolution of the system. This is formulated as the fulfilment of inequality

$$r_0/L > \lambda/r_0. \tag{40}$$

It is easy to show that condition (40) corresponds to the realisation of condition (37). In this case, the Hartmann sensor gives the correct values of the phase and the influence of amplitude fluctuations in the system can be neglected. At the same time, phase systems of adaptive optics have high efficiency.

On vertical atmospheric paths, condition (40) is easily realised, but on horizontal paths the situation changes to the opposite, i.e., the Fried coherence radius becomes smaller than the size of the first Fresnel zone:

$$r_0 < \sqrt{\lambda L}. \tag{41}$$

It turns out that the violation of condition (40) indicates the appearance of 'strong' intensity fluctuations on the path, i.e. the dispersion of intensity fluctuations calculated by the formula of S.M. Rytova [5]

$$\sigma_{\rm int}^2 = k^{7/6}C_n^2L^{11/6} \tag{42}$$

becomes higher than 1.

Under these conditions, phase systems of adaptive optics lose their effectiveness [20, 21] and it is no longer possible to achieve any improvement with the use of phase correction.

7. Conclusions

We have performed simple calculations of the intensity distribution of the image of an incoherent source object. Known errors in the evaluation of the image under short exposure have been considered. The increase in the coherence radius of the optical field has been calculated for various methods of phase correction. It is found that the emergence of 'strong' fluctuations deteriorates the efficiency of phase correction, and the wavefront sensor suffers first, i.e. with the appearance of intensity fluctuations, the focal spots will flicker apart from shifts, which greatly distorts the measurement data on the phase profile. This makes adaptive correction ineffective. At high wind speeds or when the Fried radius decreases, the correction may even lead to poor vision. Under such conditions, the effective operation of the adaptive optics system is impossible.

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