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Inverse problem of polarimetry for media with orthogonal eigenpolarisations

S.N. Savenkov, A.V. Priezzhev, E.A. Oberemok, I.S. Kolomiets, A.S. Klimov

Abstract. **We consider the problem of the ambiguity of the inverse problem of polarimetry for a homogeneous anisotropic medium characterised by orthogonal eigenpolarisations generated by non-commutativity of Jones and Mueller matrices of elementary types of anisotropy. The differences between two variants of the orthogonality conditions for the eigenpolarisations of an arbitrary anisotropic medium, the existence of which is due to this ambiguity, are investigated. The results obtained can be used to analyse the adequacy of the solution of the inverse problem of this class and in the problems of the synthesis of polarisation elements with given characteristics.**

Keywords: *inverse problem of polarimetry, Jones matrix, phase anisotropy, amplitude anisotropy, eigenpolarisations.*

1. Introduction

The most important problem of modern polarimetry is the solution of the inverse problem. The latter is defined as a problem in which the characteristics of radiation before and after interaction with a medium are partially or completely known (or can be measured) and it is necessary to extract maximum information about the medium in question.

In this paper, we turn to the class of inverse problems in which the interaction of polarised electromagnetic radiation with a medium under study does not result in depolarisation. In this class of inverse problems, the maximum possible amount of information necessary to determine the characteristics of a medium is contained in the 2×2 Jones matrix or in the 4×4 Mueller matrix [1, 2], between which there is a one-to-one correspondence [2]. In this case, the polarisation states of the radiation are described by the 2×1 Jones vector or the 4×1 Stokes vector.

In [3–5], this problem was investigated using a model of an arbitrary homogeneous anisotropic medium based on the generalised equivalence theorem, according to which [3] any combination of polarisation elements with linear and circular phase and amplitude anisotropy is equivalent to an optical system containing one element of each kind:

$$T = T^{\rm LP}(\delta,\alpha)T^{\rm CP}(\varphi)T^{\rm LA}(P,\theta)T^{\rm CA}(R), \tag{1}$$

where T is the resulting matrix of the medium under consideration. The $T^{\rm LP}$, $T^{\rm CP}$, $T^{\rm LA}$ and $T^{\rm CA}$ matrices describe four, according to Jones terminology [6], elementary types of anisotropy: linear (LP) and circular (CP) phase anisotropy [i.e. the case when electromagnetic radiation with two orthogonal linear (circular) eigenpolarisations propagates in a medium with different phase velocities], as well as linear (LA) and circular (CA) amplitude anisotropy [when radiation with two own orthogonal linear (circular) polarisations is absorbed differently]. The expressions for these matrices have the form:

$$T^{\rm LP} = \begin{pmatrix} \cos^2\alpha + \exp(-\mathrm{i}\delta)\sin^2\alpha & [1-\exp(-\mathrm{i}\delta)]\cos\alpha\sin\alpha \\ [1-\exp(-\mathrm{i}\delta)]\cos\alpha\sin\alpha & \sin^2\alpha + \exp(-\mathrm{i}\delta)\cos^2\alpha \end{pmatrix},$$

$$T^{\rm CP} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \tag{2}$$

$$T^{\rm LA} = \begin{pmatrix} \cos^2\theta + P\sin^2\theta & (1-P)\cos\theta\sin\theta \\ (1-P)\cos\theta\sin\theta & \sin^2\theta + P\cos^2\theta \end{pmatrix},$$

$$T^{\rm CA} = \begin{pmatrix} 1 & -\mathrm{i}R \\ \mathrm{i}R & 1 \end{pmatrix},$$

where δ and α are the magnitude and azimuth of the orientation of the linear phase anisotropy; φ is the magnitude of the circular phase anisotropy; P and θ are the magnitude and azimuth of the orientation of the linear amplitude anisotropy; and R is the magnitude of the circular amplitude anisotropy.

As is known, matrices of elementary types of anisotropy (2) do not commute [2]. This gives rise to one of the most important ambiguity contexts for solving the inverse problem of polarimetry for homogeneous anisotropic non-depolarising media.

Savenkov et al. [5] showed that in the general case the inverse problem under consideration has two solutions related to the existence of two different polarisation bases [orders of multiplication of matrices of elementary types of anisotropy (2)]:

$$T^{\rm CP}T^{\rm LP}T^{\rm CA}T^{\rm LA}, \tag{3a}$$

$$T^{\rm CP}T^{\rm LP}T^{\rm LA}T^{\rm CA}. \tag{3b}$$

S.N. Savenkov, E.A. Oberemok, I.S. Kolomiets, A.S. Klimov Taras Shevchenko National University of Kyiv, prosp. Glushkova 4g, 03127 Kiev, Ukraine; e-mail: sns@univ.kiev.ua;
A.V. Prizezhev Faculty of Physics and International Laser Center, M.V. Lomonosov Moscow State University, Vorob'evy gory, 119991 Moscow, Russia; e-mail: avp2@mail.ru

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Based on this result, Savenkov et al. [5] investigated how the properties (azimuths and ellipticities) of eigenpolarisations of a medium depend on the parameters of amplitude anisotropy for the case of Hermitian dichroism.

Another important problem of polarimetry in the considered context of the ambiguity of solving the inverse problem is the study of the conditions under which the eigenpolarisations of a homogeneous anisotropic medium are orthogonal. On the one hand, as can be seen from (2), all (!) four elementary types of anisotropy are characterised by orthogonal eigenpolarisations. The situation with different combinations of elementary types of anisotropy is much more complicated and diverse. Previously, Jones himself showed in the form of the so-called first equivalence theorem [6] that an arbitrary combination of polarisation elements with linear (T^{LP}) and circular (T^{CP}) phase anisotropy is always characterised by orthogonal, in the general case, elliptical eigenpolarisations. Since the Jones and Mueller matrices of circular and linear phase anisotropy are unitary, a similar theorem exists in linear algebra [7], in the sense that the product of unitary matrices is always a unitary matrix. It is noteworthy in this case that there is no such theorem for the product of Hermitian matrices, which are the matrices of linear (T^{LA}) and circular (T^{CA}) dichroism in (2). This means that in polarimetry there are no variants of the product of matrices T^{LA} and T^{CA}, which would be characterised by orthogonal eigenpolarisations.

In the class of media characterised by two elementary types of anisotropy, in the context of the orthogonality of the eigenpolarisations of a medium under study, only two cases should be mentioned. In the first case, the medium is characterised simultaneously by circular phase and circular amplitude anisotropy, which, in the framework of multiplicative simulation, corresponds to the variant of the second Jones equivalence theorem [6]. Such a medium or its equivalent combination of polarisation elements is obviously always characterised by orthogonal circular eigenpolarisations. In the second case, the medium is characterised by linear phase and amplitude anisotropy with coinciding orientation azimuths [1].

In paper [8], for the polarisation basis (3a), we obtained the conditions for the orthogonality of eigenpolarisations of a medium in the general case, i. e., when the medium under study is characterised by all four elementary types of anisotropy. The aim of this paper is to obtain similar general orthogonality conditions for eigenpolarisations of an arbitrary anisotropic medium for the polarisation basis (3b) and to perform a comparative analysis of these conditions.

2. Conditions for the orthogonality of an arbitrary anisotropic medium

As shown in [8], for the Jones matrix model in basis (3a) the conditions for orthogonality of the eigenpolarisations for an arbitrary homogeneous anisotropic medium have the form

$$\begin{aligned}&(1-P)\{(1+R^2)\cos[2(\alpha-\theta-\varphi)]\\&\quad-(1-R^2)\cos[2(\alpha-\theta)]\}=0,\\&2R\tan\left(\frac{\delta}{2}\right)=\left(\frac{1-P}{1+P}\right)\{(1-R^2)\sin[2(\alpha-\theta)]-\\&\quad-(1+R^2)\sin[2(\alpha-\theta-\varphi)]\}.\end{aligned}\tag{4}$$

Using arguments and actions similar to those given in [8], we obtain the conditions for the orthogonality of eigenpolarisations for basis (3b):

$$\begin{aligned}&(1-P)\{(1+R^2)\cos[2(\alpha-\theta)]\\&\quad-(1-R^2)\cos[2(\alpha-\theta-\varphi)]\}=0,\\&2R\tan\left(\frac{\delta}{2}\right)=\left(\frac{1-P}{1+P}\right)\{(1+R^2)\sin[2(\alpha-\theta)]\\&\quad-(1-R^2)\sin[2(\alpha-\theta-\varphi)]\}.\end{aligned}\tag{5}$$

It can be seen that orthogonality conditions (4) and (5) are structurally very similar, and their differences are as follows: different arguments of cosines in the first equations (where there is a quantity $-\varphi$ in one basis, which is absent in another basis) and different signs in front of R^2 in the second equations.

3. Comparative analysis of the obtained solutions

Further analysis will be carried out for two cases: $\alpha=\theta$ and $\alpha\neq\theta$, i. e., the equality and inequality of azimuths of orientation of the linear phase and amplitude anisotropy.

3.1. Case $\alpha=\theta$

In this case, the systems of equations (4) and (5) take the form

$$\begin{aligned}&(1-P)[-1+R^2+(1+R^2)\cos 2\varphi]=0,\\&-\frac{(-1+P)(1+R^2)\cos\varphi\sin\varphi}{1+P}-R\tan\left(\frac{\delta}{2}\right)=0,\end{aligned}\tag{6a}$$

$$\begin{aligned}&-(1+P)[1+R^2+(-1+R^2)\cos 2\varphi]=0,\\&\frac{(-1+P)(-1+R^2)\cos\varphi\sin\varphi}{1+P}-R\tan\left(\frac{\delta}{2}\right)=0.\end{aligned}\tag{6b}$$

The general (in the sense that the medium in question is simultaneously characterised by all four elementary types of anisotropy) solution of system (6a) with respect to the parameters of phase anisotropy was obtained in [9] and has the form:

$$\begin{aligned}&\delta=\pm 2\arctan\left(\frac{|P-1|}{1+P}\right),\\&\varphi=\pm\arccos\left(\mp\frac{1}{\sqrt{1+R^2}}\right).\end{aligned}\tag{7}$$

At the same time, the analysis of system (6b), which is similar to the one we performed for system (6a) in [9], shows there is no such general solution for it.

As an example illustrating this result, we consider the solution of the inverse problem in bases (3a) and (3b) for the following Jones matrix:

$$T = \begin{pmatrix} 0.96 - i0.2 & -0.23 - i0.47 \\ 0.36 + i0.37 & 0.6 + i0.05 \end{pmatrix}. \tag{8}$$

In basis (3a) we have $\delta = -20°$, $\alpha = 10°$, $\varphi = 26.6°$, $P = 0.7$, $\theta = 10°$, and $R = 0.5$, and in basis (3b) we have $\delta = -31.3°$, $\alpha = -14.4°$, $\varphi = -24.4°$, $P = 0.56$, $\theta = 10°$, and $R = 0.45$. Thus, in the first case, $\alpha = \theta$ and the existing solution corresponds to conditions (7). In the second case, a solution also exists, but in this case, $\alpha \neq \theta$.

For both systems, (6a) and (6b), there are no solutions in the case when the medium under study is simultaneously characterised by any three elementary types of anisotropy from (2).

Now we turn to conditions (7). It is seen that when the orientation azimuths of the phase and amplitude anisotropy coincide, the value of phase linear anisotropy δ is completely determined only by the value of the parameter P, i.e. linear amplitude anisotropy, and the value of the circular phase anisotropy φ is completely determined only by the value of the parameter R, i.e. circular amplitude anisotropy. Graphic interpretation of conditions (7) is presented in Fig. 1.

Important in this case is that, as follows from Fig. 1a, the magnitude of linear phase anisotropy δ, when the magnitude of linear amplitude anisotropy P varies in the whole range of its physically acceptable values, can reach values only in the range of 0 to 90°, the values of δ that are greater than 90° cannot be obtained. It follows from Fig. 1b that with a change in the parameter R in the whole range of physically acceptable values, the value of φ can change only in the range of 0 of 45° and of 135° to 180°. Thus, the parameters δ and φ in the case of the orthogonality of eigenpolarisations have 'forbidden zones' of values in the range of 90° to 180° and of 45° to 135°, respectively. It should be noted that the widths of the forbidden zones are equal to π in both cases.

Note that conditions (6a) and (6b) obviously describe all media characterised by two types of anisotropy, which were discussed in Section 1, including those media that are characterised by linear phase and amplitude anisotropy with coincident orientation azimuths.

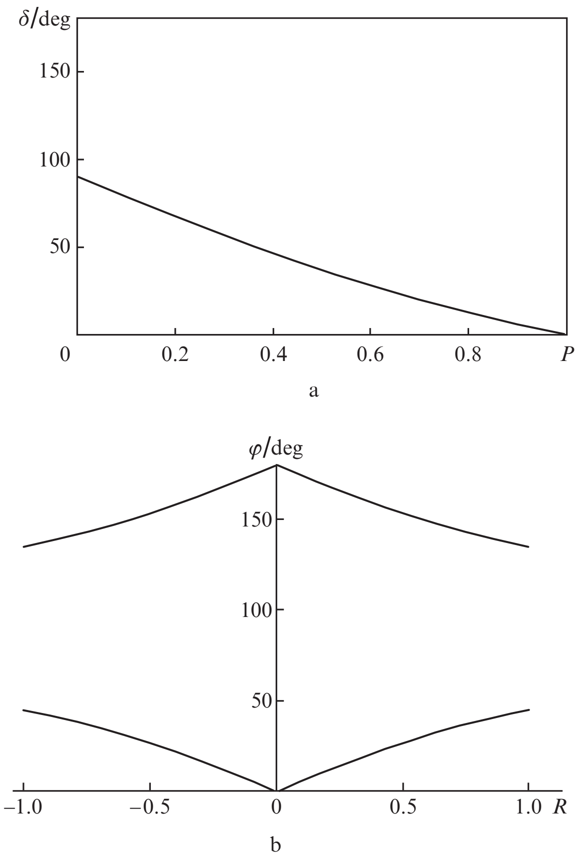

Figure 1. Orthogonality conditions for eigenpolarisations (7): (a) linear phase anisotropy δ as a function of linear amplitude anisotropy P and (b) circular phase anisotropy φ as a function of circular amplitude anisotropy R.

3.2. Case $\alpha \neq \theta$

In this case, the general solutions of systems (4) and (5) take respectively the form:

$$\varphi = \Delta \mp \frac{1}{2}\arccos\left[-\frac{(R^2-1)\cos 2\Delta}{R^2+1}\right],$$

$$\delta = \tag{9}$$

$$2\arctan\left\{\frac{(P-1)\{(R^2-1)\sin 2\Delta + (R^2+1)\sin[2(\Delta-\varphi)]\}}{2R(P+1)}\right\},$$

$$\varphi = \Delta \mp \frac{1}{2}\arccos\left[-\frac{(R^2+1)\cos 2\Delta}{R^2-1}\right],$$

$$\delta = \tag{10}$$

$$-2\arctan\left\{\frac{(P-1)\{(R^2+1)\sin 2\Delta + (R^2-1)\sin[2(\Delta-\varphi)]\}}{2R(P+1)}\right\},$$

where $\Delta = \alpha - \theta$.

Thus, in the case of $\alpha \neq \theta$ in both bases, (3a) and (3b), there are general solutions to the inverse problem, which are represented by the systems of equations (9) and (10). These solutions differ in sign between R^2 and 1, as well as the sign of the parameter δ. The graphical interpretation of conditions (9) and (10) is presented in Figs 2 and 3. It can be seen that for basis (3a) the values of parameters φ and δ are physically acceptable for any values of a pair of parameters R and Δ, whereas for basis (3b) there are forbidden zones for the values of the parameters φ and δ, as in the case of $\alpha = \theta$.

We should add that systems (4) and (5) in the case when the medium under study is simultaneously characterised by any three elementary types of anisotropy from (2) have the only solutions [4, 5]

$$\varphi = 0,\ \pi,$$

$$\alpha = \theta \pm \frac{\pi}{4}, \tag{11}$$

$$\delta = 2\arctan\left\{\frac{R[\mp(1-P)]}{1+P}\right\},$$

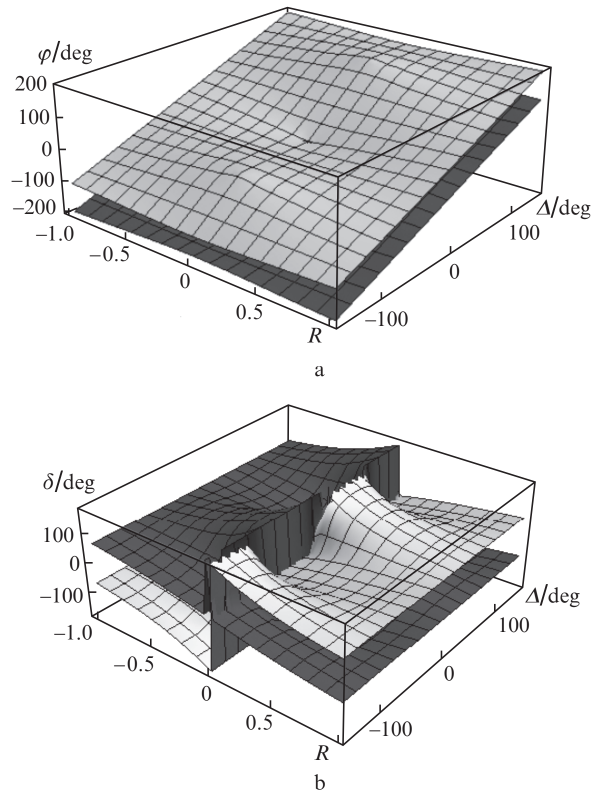

Figure 2. Orthogonality conditions (9): dependences of the magnitude of (a) circular phase anisotropy φ and (b) linear phase anisotropy δ on the magnitude of the circular amplitude anisotropy R and $\Delta = \alpha - \theta$ at $P = 0.2$.

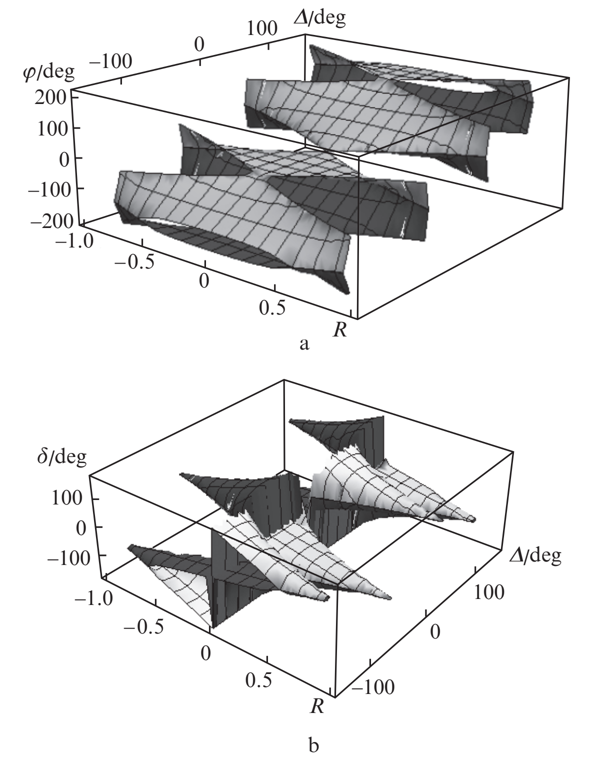

Figure 3. Orthogonality conditions (10): dependences of the magnitude of (a) circular phase anisotropy φ and (b) linear phase anisotropy δ on the magnitude of the circular amplitude anisotropy R and $\Delta = \alpha - \theta$ at $P = 0.2$.

$$\varphi = 0, \pi,$$
$$\alpha = \theta \pm \frac{\pi}{4}, \qquad (12)$$
$$\delta = 2\arctan\left\{\frac{R[\pm(1-P)]}{1+P}\right\}.$$

These solutions correspond to the so-called Hermitian dichroism and can be considered as dichroic polar forms in an anisotropic medium model based on the polar decomposition theorem [10].

4. Conclusions

We have considered the problem of the ambiguity of the inverse problem of polarimetry for a homogeneous anisotropic medium characterised by orthogonal eigenpolarisations generated by the non-commutativity of Jones and Mueller matrices of elementary types of anisotropy (2). The non-commutativity of Jones and Mueller matrices leads to the presence of two solutions of the inverse problem of polarimetry, related to the existence of two different polarisation bases (3a) and (3b) [5].

For the polarisation basis (3b), conditions (5) have been found under which an arbitrary anisotropic medium is characterised by orthogonal eigenpolarisations. For conditions (5) and conditions (4), which we obtained in [8], a comparative analysis has been performed.

For the case $\alpha = \theta$, it has been shown that there is no general solution for conditions (5). At the same time, for conditions (4), the solution exists [9]. This means that if for a medium with orthogonal eigenpolarisations, whose anisotropic properties are described by some Jones matrix, within the polarisation basis (3a) there is a solution to the inverse problem containing all four elementary types of anisotropy, then within basis (3b) there is always (!) a solution for this Jones matrix, containing only three elementary types of anisotropy.

It has also been shown that in the case $\alpha = \theta$, there are no solutions to the inverse problem of the class under consideration that simultaneously contain three elementary types of anisotropy in bases (3a) and (3b).

General solutions have been obtained for the inverse problem (9) and (10) for the case $\alpha \neq \theta$. The characteristic features of these solutions are that, if we consider the dichroic parameters P and R to be arbitrary, the magnitude of the circular phase anisotropy φ depends only on R and the difference Δ of orientation azimuths of the linear phase and amplitude anisotropy. The magnitude of the linear phase anisotropy δ depends on all the two above parameters and on P. However, it has been shown that in basis (3a), physically acceptable values of the parameters φ and δ can be found for any pair of the values of the parameters R and Δ, whereas in basis (3b), such values of φ and δ exist not for every pair of the parameters R and Δ.

The results obtained in the work, on the one hand, are of undoubted interest for an adequate interpretation of the polarimetric measurement data, and on the other hand, they can be used in the synthesis of polarisation elements with given characteristics.

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