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Thermodynamically equilibrium quantum vortices in superfluid liquids

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Abstract. **The thermodynamic equilibrium is studied for a system of quantised vortices in superfluid liquids in the presence of a counterflow of the normal and superfluid components. The partition function is calculated, which takes into account various configurations of the vortex filaments, as well as the distribution of loops in length. The dependence of the density of vortex filaments on the applied counterflow velocity is discussed.**

Keywords: *quantum vortices, superfluid liquid, Gibbs distribution, partition function.*

1. Introduction

In our previous paper [1], we discussed chaotic vortex filaments, or quantum turbulence, in superfluid liquids and in Bose–Einstein condensate (BEC) excited by the counterflow of the normal and superfluid components. One of the discussed aspects of the quantum turbulence theory was the extraordinary complexity of this problem from the point of view of theoretical study and the need for some particular approach to the general problem. An important approach is the study of thermodynamic equilibrium in a system of quantised vortices in superfluid helium and in BEC in the case of a counterflow of the normal and superfluid components. It should be emphasised that this problem is important and interesting in itself, but from the point of view of turbulence, it is also necessary to clarify a number of aspects of the structure and dynamics of a vortex tangle.

2. Statement of the problem. The discussion of the results

This paper describes the dynamics of a vortex filament under the action of a random Langevin force in superfluid helium and in BEC in the presence of a counterflow with a relative velocity $\boldsymbol{v}_{\rm ns} = \boldsymbol{v}_{\rm n} - \boldsymbol{v}_{\rm s}$. This formulation of the problem is motivated by the fact that ordinary quantum turbulence develops in a counterflow of superfluid helium without random mixing due to the development of instabilities. Therefore, it is important to compare both mechanisms of generation of a vortex tangle. Let us choose the equation of motion of the elements of the vortex line in superfluid helium in the form (see, for example, [2])

$$\dot{\boldsymbol{s}}(\xi,t) = \dot{\boldsymbol{s}}_{\rm i}(\xi,t) + \boldsymbol{v}_{\rm s} + \alpha\dot{\boldsymbol{s}}(\xi,t)\times(\boldsymbol{v}_{\rm n} - \boldsymbol{v}_{\rm s} - \dot{\boldsymbol{s}}_{\rm i}(\xi,t)) + \boldsymbol{\zeta}(\xi,t). \quad (1)$$

Here, $\boldsymbol{s}(\xi,t)$ are the radius vectors of the elements of the vortex line; $\dot{\boldsymbol{s}}_{\rm i}(\xi,t)$ is the self-induced velocity; and α is the friction coefficient. The quantity $\dot{\boldsymbol{s}}_{\rm i}(\xi,t)$ is related to the geometric shape of the line [2]. The Langevin force $\boldsymbol{\zeta}(\xi,t)$ is introduced into the equation of motion of the filament elements (1), which simulates the thermal effect from the thermostat.

Vortices in BEC are usually examined based on the study of a macroscopic wave function that obeys the nonlinear Schrödinger equation. The historical aspects of the discovery and study of BEC in ultracold atomic gases are described, for example, in reviews [3, 4]. The theoretical problems of the BEC dynamics are presented in the well-known book of Pitaevskii and Stringari [5]. In the domestic literature, relevant studies (both theoretical and experimental) are described in Refs [6–8].

For the centre line of the vortex in BEC, one can obtain an equation similar to relation (1) from [9]. We confine ourselves to the study of thermodynamic equilibrium; therefore, we assume that the correlation function for the Langevin force $\boldsymbol{\zeta}(\xi,t)$ satisfies the fluctuation–dissipation theorem [10, 11]:

$$\langle \zeta_\alpha(\xi_1,t)\zeta_\beta(\xi_2,t')\rangle = \frac{k_{\rm B}T\alpha}{\rho_{\rm s}\kappa}\delta(\xi_1-\xi_2)\delta(t_1-t_2)\delta_{\eta_1\eta_2}. \quad (2)$$

The Fokker–Planck equation for the time evolution of the functional of the probability distribution $P(\{\boldsymbol{s}(\xi)\},t) = \langle\delta(\boldsymbol{s}(\xi) - \boldsymbol{s}(\xi,t))\rangle$ can be obtained from the equation of motion (1) in a standard way (see, for example., [11]):

$$\frac{\partial P}{\partial t} + \int {\rm d}\xi\frac{\delta}{\delta\boldsymbol{s}(\xi)}\{[\dot{\boldsymbol{s}}_{\rm i}(\xi) + \boldsymbol{v}_{\rm s} + \alpha\boldsymbol{s}'(\xi)\times(\boldsymbol{v}_{\rm n} - \boldsymbol{v}_{\rm s} - \dot{\boldsymbol{s}}_{\rm i}(\xi))]P\}$$
$$+\iint {\rm d}\xi{\rm d}\xi'\,\langle\zeta_\alpha(\xi)\zeta_\beta(\xi')\rangle\,\delta(\xi-\xi')\delta(t_1-t_2)\delta_{\eta_1\eta_2}$$
$$\times\frac{\delta}{\delta\boldsymbol{s}_{\eta_1}(\xi)}\frac{\delta}{\delta\boldsymbol{s}_{\eta_2}(\xi')}P = 0. \quad (3)$$

It was shown in [12] that equation (3) has a solution in the form of the Gibbs distribution:

$$P(\{\boldsymbol{s}(\xi)\},t) - N\exp\left(-\frac{H\{\boldsymbol{s}\}}{k_{\rm B}T}\right). \quad (4)$$

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Here, N is the normalisation factor and $\beta = 1/k_B T$. The Hamiltonian $H\{\boldsymbol{s}\}$ with a nonzero relative velocity $\boldsymbol{v}_{ns}$ has the form

$$H\{\boldsymbol{s}\} = E\{\boldsymbol{s}\} - \boldsymbol{P}(\boldsymbol{v}_n - \boldsymbol{v}_s). \tag{5}$$

Here the energy $E\{\boldsymbol{s}\}$ and the Lamb momentum $\boldsymbol{P}\{\boldsymbol{s}\}$ are defined (see, for example, [13]) as

$$E\{\boldsymbol{s}\} = \frac{\rho_s \kappa^2}{8\pi} \int_\Gamma \int_{\Gamma'} \frac{\boldsymbol{s}'(\xi)\cdot \boldsymbol{s}'(\xi')}{|\boldsymbol{s}(\xi) - \boldsymbol{s}(\xi')|} \mathrm{d}\xi \mathrm{d}\xi',$$
$$\boldsymbol{P}\{\boldsymbol{s}\} = \frac{\rho_s \kappa}{2} \int \boldsymbol{s}(\xi) \times \boldsymbol{s}'(\xi) \mathrm{d}\xi. \tag{6}$$

In ordinary statistical mechanics, if $\boldsymbol{P}$ is the true momentum of a particle (or a quasi-particle), the validity of equations (4) and (5) is obvious and follows from Galilean transformations. Since, for vortices, the Lamb momentum $\boldsymbol{P}$ is not a 'real' pulse, the validity of relations (4) and (5) is obtained from the exact solution of the Fokker–Planck equation (3).

Relations (1)–(3) should be used to calculate the partition function and, accordingly, to determine the various properties of the vortex tangle. By definition [14, 15] the partition function has the form

$$Z(T, \boldsymbol{v}_{ns}) = \sum_{\{s_j(\xi_j)\}} \exp\Big(-\frac{H\{\boldsymbol{s}\}}{k_B T}\Big). \tag{7}$$

Here the expression

$$\{\boldsymbol{s}(\xi)\} = \bigcup_j \boldsymbol{s}_j(\xi_j)$$

is a set of different loops. The sum symbol implies a contribution from all possible configurations of each of the loops of length l, summation over all loops with different lengths l, and integration over the starting points of each loop. Of course, it is not possible to list all continuous curves. Usually some discrete analogues are used, for example, lattice models or a polymer chain model (or their combination), in which the number of line configurations increases with its length as $\exp[C(l/a)]$. Here, the constant $C \sim 1$, its exact value is determined by the used model. For example, for a three-dimensional cubic lattice, $C = \ln(2D - 1)$ (D is the dimension of space). The value of a is the parameter of the model length. For a model with a cubic lattice, this is a cube edge, and for a polymer it is an elementary step. In the case of quantum vortices, the quantity a coincides with the coherence length and can be taken as the size of the radius of the vortex core (for more details, see [14, 15]).

The computed total number of different configurations, $\exp[C(l/a)]$, should be limited to the choice of specific configurations that fit our task. For example, if we consider closed loops of length l, then the total number of configurations $\exp[C(l/a)]$ must be multiplied by the probability $p(l)$ of obtaining this configuration. The latter problem relates to the physics of polymers, and therefore, the vortex filaments possess the topology of polymer chains. It is known that the probability $p(l)$ can be written in the form of an integral over trajectories in the following manner (see, for example, [16]):

$$p(l) = N\Big(\frac{a}{l}\Big) \int_{s(0)=r}^{s(l)=r} \mathcal{D}\boldsymbol{s}(\xi) \exp\Big[-\frac{3}{2a} \int (\boldsymbol{s}'(\xi))^2 \mathrm{d}\xi\Big]. \tag{8}$$

Here, the quantity a/l eliminates the degeneracy associated with the choice of starting points on the loop in the functional integral. The Gibbs factor has the form

$$\exp\Big[-\beta w \frac{\rho_s \kappa^2}{8\pi} \iint \frac{\boldsymbol{s}'(\xi_1)\cdot \boldsymbol{s}'(\xi_2)}{|\boldsymbol{s}(\xi_1) - \boldsymbol{s}(\xi_2)|} \mathrm{d}\xi_1 \mathrm{d}\xi_2 + \beta u \boldsymbol{v}_{ns} \frac{\rho_s \kappa}{2} \int \boldsymbol{s}(\xi) \times \boldsymbol{s}'(\xi) \mathrm{d}\xi\Big]. \tag{9}$$

The parameters u and w are introduced for convenience. Below we will use u and w equal to unity ($u = 1$, $w = 1$).

In addition to the configuration factor (8) and the Gibbs factor (9), it is necessary to add summation over all loops of different lengths l and integration over the initial points $\boldsymbol{r}_{start}$ of each loop, i.e.

$$\int \mathrm{d}\boldsymbol{r}_{start} \int n(l) \mathrm{d}l. \tag{10}$$

In one important case, the so-called local approximation case, the energy can be expressed as $E = \varepsilon_V l$ (here, ε_V is the energy per unit length). Then the configuration factor (8) and the Gibbs factor (9) are combined as follows:

$$\int \mathcal{D}\boldsymbol{s}(\xi) \exp\Big[-\frac{3}{2a} \int (\boldsymbol{s}'(\xi))^2 \mathrm{d}\xi - \beta u \boldsymbol{v}_{ns} \frac{\rho_s \kappa}{2} \int \boldsymbol{s}(\xi) \times \boldsymbol{s}'(\xi) \mathrm{d}\xi\Big] \times \exp\Big[-\beta\Big(w\varepsilon_V - \frac{C}{\beta a}\Big) l\Big]. \tag{11}$$

It can be seen that for such independent strings the partition function diverges at a certain temperature and there occurs a phase transition. The corresponding temperature T_H is called the Hagedorn temperature [14, 15] and is defined as

$$T_H = \frac{\varepsilon_V a}{k_B C}. \tag{12}$$

The continuum integral (11) is Gaussian, and the partition function can be calculated exactly. This problem coincides with the problem of the motion of a charged particle in a constant magnetic field [15]. Performing the procedures described in [15], we arrive at the expression for the partition function:

$$Z_l = \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}l \Big\{ n(l) \frac{a}{l} \Big(\frac{3}{2\pi a l}\Big)^{3/2} \Big[\frac{\sin(l u \kappa \beta a \rho_s v_{ns}/6)}{l u \kappa \beta a \rho_s v_{ns}/6}\Big]^{-1} \times \exp\Big[-\beta\Big(w\varepsilon_V - \frac{\ln 5}{\beta a}\Big) l\Big]\Big\}. \tag{13}$$

Using (13), we determine the average energy and momentum:

$$\langle E\{\boldsymbol{s}\}\rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial w}\Big|_{u=1, w=1}, \quad \langle \boldsymbol{P}\cdot \boldsymbol{v}_{ns}\rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial u}\Big|_{u=1, w=1}. \tag{14}$$

Given that in the local approximation the energy is proportional to the length, these formulae make it possible to find the density of vortex filaments or the total length per unit volume. Using the obtained partition function, we calculate the structural factors of quantum turbulence, for example, the average polarisation of the vortex loops that make up the vortex tangle in a counterflow of He II, as well as the anisotropy and average curvature. These factors of quantum turbulence were previously obtained only numerically in [2]. It is interesting to compare the data on the equilibrium properties of a vortex tangle, which can be obtained on the basis of the formalism developed here, with those on quantum turbulence.

We describe some physical effects that follow directly from relation (13). Expanding the expression in square brackets in degrees of the relative velocity $\boldsymbol{v}_{\rm ns}$ and using only the terms of the second order, we obtain

$$\left[\frac{\sin(lu\kappa\beta a\rho_{\rm s}v_{\rm ns}/6)}{lu\kappa\beta a\rho_{\rm s}v_{\rm ns}/6}\right]^{-1} = 1 - \frac{1}{216}a^2l^2u^2v_{\rm ns}^2\kappa^2\beta^2\rho_{\rm s}^2 + O(v_{\rm ns}^3). \quad (15)$$

Thus, the partition function contains a term independent of the relative velocity $\boldsymbol{v}_{ns}$, as well as a term proportional to the square of this quantity. The first term describes simply thermodynamically equilibrium vortices in helium at rest. The problem of thermodynamic quantum vortices in helium at rest was previously investigated under various assumptions. The most popular were the studies in which vortex filaments were considered to be ideal rings (see, for example, [17–19]). Another direction, a numerical study of similar problems, was to perform calculations for topological defects (see, for example, [20]). Quantum vortices in BEC can be studied by using a macroscopic wave function obeying the nonlinear Schrödinger equation. The analysis is reduced to an elegant, albeit complex mathematical apparatus for studying the zeros of a fluctuating macroscopic wave function. Recall that the vortex lines in BEC are a locus of points at which the wave function vanishes. Examples of such studies can be work [21, 22]. In these papers the following results were obtained. The density L of the vortex filaments at the Hagedorn temperature $T_{\rm H}$ is a value of the order of the coherence length, that is, $L \sim (1/a)^2$. Then, L exponentially decreases with decreasing temperature. Formulae (13) and (14) show that our results are in qualitative agreement with those presented above.

Of particular interest is the second term on the right-hand side of formula (5), which contains the relative velocity $\boldsymbol{v}_{\rm ns} = \boldsymbol{v}_{\rm n} - \boldsymbol{v}_{\rm s}$. This term is an order of magnitude smaller than the first term; therefore, the density of vortices is much lower. The structure of the corresponding set of vortex loops is very close to what is called superfluid turbulence. An important factor is that the density of the vortex lines in this case depends on the square of the relative velocity: $L \propto \boldsymbol{v}_{\rm ns}^2$. This is a well established experimentally and numerically fact (see, for example, [23, 24]). According to our data, no theoretical methods for obtaining this dependence are still known. Our result has been obtained, however, for the thermodynamically equilibrium case, and it is not yet clear how it relates to the case of quantum turbulence. This question, as well as other questions concerning the relationship of thermodynamic equilibrium with a turbulent flow, is of great interest and will be investigated in the future.

3. Conclusions

Using the Langevin formulation of the problem, the dynamics of a vortex filament is described in terms of the thermodynamic equilibrium of the vortex system. The corresponding Gibbs distribution depends on both the energy and the momentum of the vortex loops. The partition function is calculated, which includes the configurational distribution of the vortex lines. A preliminary analysis of the partition function leads to the conclusion that there are two types of vortex filaments. These are thermodynamic vortices generated by thermal fluctuations and hydrodynamic vortices associated with the presence of a counterflow.

In their structure, hydrodynamic vortices resemble a vortex tangle, which is observed in the case of quantum turbulence.

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