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Quantum geometric phase under pre- and post-selection

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Abstract. **We consider a quantum system subjected to a controlled phase transformation and interaction with the environment in between the acts of selection, which leads to the emergence of interference effects. It is shown that the shift of the dependence of the statistics of contacts' information with the environement on the controlled phase shift can be interpreted as a geometric phase. This interpretation is consistent with the known operational approach to the geometric phase. As a result, we suggest generalising the operational approach to the realm of pre- and post-selected quantum states.**

Keywords: *quantum geometric phase, Mach–Zehnder interferometer, two-state vector formalism.*

1. Introduction

Any pure state of a quantum system can be represented by a point in a complex projective space. Therefore, continuous evolution of a pure quantum state of an isolated system can be presented in the form of a continuous curve C in this space, to which a complex number $\exp[i\varphi_{\rm geom}(C)]$ is related. Its argument $\varphi_{\rm geom}(C)$ is known as geometric (topological) phase [1]. It is a kinematic notion stipulated solely by the shape of C and independent of the speed with which the system sweeps along the curve. The modern concept of geometric phase stems from works [2, 3], though some of its optical manifestations were revealed earlier [4]. The development and numerous applications of geometric phase in quantum and classical physics are outlined in [5, 6].

This initial concept of a quantum geometric phase has undergone several generalisations. Thus, Uhlmann [7] proposed the first geometric phase generalisation to mixed quantum states. The physically oriented treatment based on an operational definition of the mixed state geometric phase under unitary evolution was implemented in [8]. Since the geometric phase can be measured in interferometric experiments, the authors of Ref. [8] suggested using the Mach–Zehnder (MZ) scheme in the following manner: the inner state of the system undergoes unitary transformation on one of the interferometer's arms, while a phase shift θ is inserted on the second arm. The probability difference to detect the system in one or another output channel of the interferometer demonstrates the interference dependence $\cos(\theta - \theta_{\rm geom})$ on θ. The shift of the cosine argument gets interpreted as a geometric phase. The $\theta_{\rm geom}$ depends on the unitary transformation and the input state, and in the case of the pure one, retrieves the known definition of geometric phase. In this scheme, the quantum system is required to be able to propagate along the superposition of various spatial trajectories in order to obtain the geometric phase. A relevant example of such a system is a photon with a polarisation as its inner state.

The next natural significant step in developing the theory of geometric phase is its definition for an open quantum system undergoing nonunitary evolution. It turns out that generalisation of the geometric phase concept to such systems depends heavily on the chosen approach (see [9–11] for details). In the context of the present work, the approach studied in [12] is of particular importance. In a sense, it unifies ideas from [8, 10, 11]. Here, the geometric phase is detected in an interferometric measurement scheme within the aforementioned operational approach. The unitary evolution from [8] is replaced by the interaction of the system with an element of environment followed by the measurement of some environmental observable. Consequently, the quantum system under investigation becomes open due to its informational contact with the environment. It results in a nonunitary transformation of the internal state of the system. In [12] the idea of a special geometric phase stipulated by the measurement outcomes in the interferometric scheme was proposed. This conjecture has been verified in [13]. For the case of a qubit system travelling along a sequence of Mach–Zehnder interferometers, there was obtained a general expression for the geometric phase as a function of the history of qubit's contacts with the environment, i.e. a sequence of fixed measurement outcomes upon the system's passing through a chain of interferometers, given an arbitrary input state.

The aim of the present work is to extend the operational notion of geometric phase into the realm of the so-termed "two-state quantum physics" known also as "time-symmetric quantum physics" [14]. We exploit the approach which incorporates significant features of those from [8, 12]. Upon application to the time-symmetric setting, the approach reveals new peculiarities of the latter. At the same time, it provides a natural step toward the revealing of the geometric aspects of quantum evolution in the case of the most general form of the system's state known up today.

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2. Scheme for detecting the geometric phase

The scheme introducing geometric phase under pre- and post-selection is a modification of the interferometric arrangements used in [8, 12]. In one arm of a MZ-interferometer (see Fig. 1) a controllable phase shift θ is inserted. In the second arm, the system interacts with a standard element of the environment, initially prepared in the state $|e\rangle$ via the unitary operator $\hat{U}$. Upon a measurement, the standard element finally can be found in an eigenstate $|e_i\rangle$ of the measured observable. The prearranged state $|\Psi\rangle^{\rm pre}$ of the system entering the interferometer is the result of successful preparation, i.e. pre-selection. Similarly, experimental runs with post-selection of the system in $|\Psi\rangle^{\rm post}$ are only considered as successful. As mentioned, the probabilities to find the system at one of the MZ exits are used as interference fringes. Since the output of the interferometer is post-selected, the probabilities of various measurement outcomes are interpreted as “interference fringes”.

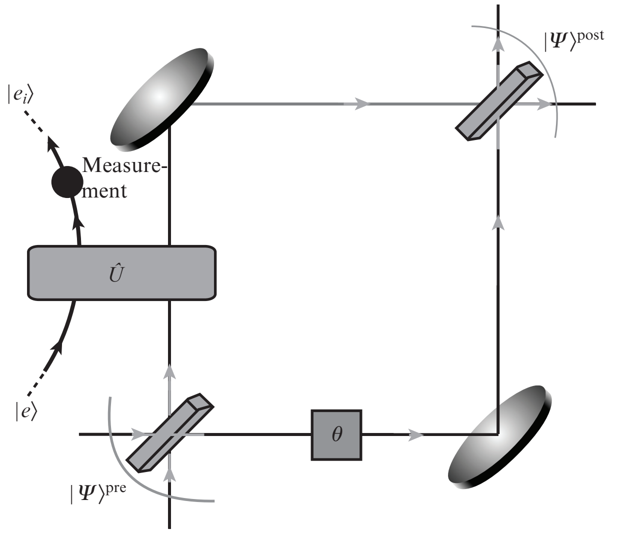

Figure 1. Scheme of the MZ interferometer with selection of the system’s states before and after its passing through the interferometer.

Following notations from [8, 12, 13], $|0\rangle$ and $|1\rangle$ stand for the system’s presence in horizontal and vertical internal or external arms of the interferometer. Input and output beam splitters make the following transformations:

$$|0\rangle\to\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle),\quad |1\rangle\to\frac{1}{\sqrt{2}}(-|0\rangle+|1\rangle). \tag{1}$$

The folding mirrors are responsible for the transformations $|0\rangle\to|1\rangle$, $|1\rangle\to|0\rangle$. We assume pre- and post-selection of pure states to have the form:

$$|\Psi\rangle^{\rm pre}=|\psi_0^{\rm pre}\rangle\otimes|0\rangle+|\psi_1^{\rm pre}\rangle\otimes|1\rangle,$$

$$|\Psi\rangle^{\rm post}=|\psi_0^{\rm post}\rangle\otimes|0\rangle+|\psi_1^{\rm post}\rangle\otimes|1\rangle. \tag{2}$$

Here $|\psi_\sigma^{\rm pre}\rangle$ and $|\psi_\sigma^{\rm post}\rangle$ ($\sigma=0,1$) are the vectors in the Hilbert space $\mathcal{H}_{\rm s}$ of internal states of the system. The effect of interaction between the system and the environment’s element on their compound state is given by a unitary operator $\hat{U}$. Depending on two alternative directions from which the system enters the interferometer, the ith possible outcome of the measurement results in two possible transformations of the system state :

$$|\psi\rangle\otimes|0\rangle\to\langle e_i|e\rangle|\psi\rangle\otimes|0\rangle\equiv(\mathcal{E}_i(0)|\psi\rangle)\otimes|0\rangle,$$
$$|\psi\rangle\otimes|1\rangle\to\langle e_i|\hat{U}|e\rangle|\psi\rangle\otimes|1\rangle\equiv(\hat{\mathcal{E}}_i(1)|\psi\rangle)\otimes|1\rangle. \tag{3}$$

The first transformation is merely a complex factor $\mathcal{E}_i(0)$ acquired by the system’s internal state $|\psi\rangle$ since in fact no real interaction takes place. The second one introduces a nontrivial transformation $\hat{\mathcal{E}}_i(1)$ as a consequence of interaction and subsequent reading of the measurement outcome i.

We assume a qubit-type of the environment’s element. This allows us to deal with the simplest version of the operational approach since the interference phenomenon may be looked for in the probability difference $p_1(\theta)-p_2(\theta)$ of two alternative outcomes. The probability amplitude $A_i(\theta)$ to get the outcome i and than to witness successful post-selection of the system in the $|\Psi\rangle^{\rm post}$ * can be written as:

$$A_i(\theta)=\langle\psi_+^{\rm post}|\hat{\mathcal{E}}_i(1)|\psi_+^{\rm pre}\rangle-\exp(\mathrm{i}\theta)\mathcal{E}_i(0)\langle\psi_-^{\rm post}|\psi_-^{\rm pre}\rangle. \tag{4}$$

Here $|\psi_\pm^{\rm pre,\,post}\rangle=(|\psi_0^{\rm pre,\,post}\rangle\pm|\psi_1^{\rm pre,\,post}\rangle)/\sqrt{2}$. The amplitudes $A_i(\theta)$ by means of the Aharonov–Bergmann–Lebowitz (ABL) rule [15] give the probabilities $p_i(\theta)$ to get the ith outcome under the condition of successful pre- and post-selections. For the probability difference we have

$$p_1(\theta)-p_2(\theta)=\frac{|A_1(\theta)|^2-|A_2(\theta)|^2}{|A_1(\theta)|^2+|A_2(\theta)|^2}. \tag{5}$$

The dependence on θ is given by a term $\cos(\theta-\theta_{\rm geom}^-)$ in its numerator and a similar term $\cos(\theta-\theta_{\rm geom}^+)$ in its denominator. Expressions for the two geometric phase shifts have the form:

$$\theta_{\rm geom}^\pm=\arg[\langle\psi_+^{\rm post}|(\hat{\mathcal{E}}_0(1)\mathcal{E}_0^*(0)$$
$$\pm\hat{\mathcal{E}}_1(1)\mathcal{E}_1^*(0))|\psi_+^{\rm pre}\rangle\langle\psi_-^{\rm pre}|\psi_-^{\rm post}\rangle]. \tag{6}$$

The appearance of two geometric phases instead of a single one in previous applications stems evidently from peculiarities of the ABL rule for retrospectively calculating probabilities of measurement outcomes preceding the posts-selection. The ABL rule is a variant of ‘Bayesian inference’ and differs immensely from the usual quantum expression for probabilities in the case of pre-selected systems.

The phases $\theta_{\rm geom}^\pm$ can be understood as sums of two terms, i.e. the arguments of multipliers in the square brackets in (6). It is the combination of these two terms which guarantees the gauge invariance as the main property of geometric phase. Indeed, any transformation $|\Psi\rangle^{\rm pre}\to\exp(\mathrm{i}\phi)|\Psi\rangle^{\rm pre}$ and $|\Psi\rangle^{\rm post}\to\exp(\mathrm{i}\phi')|\Psi\rangle^{\rm post}$ does not change $\theta_{\rm geom}^\pm$. Only the first multiplier in the right-hand side of (6) depends on the measurement type. In the limit of negligibly weak interaction between the system and environment, $\hat{U}$ tends to unit operator so that $\hat{\mathcal{E}}_i(1)$ becomes equal to $\hat{\mathcal{E}}_i(0)$ and

$$\theta_{\rm geom}^\pm\to\theta_{\rm geom}\doteq\arg(\langle\psi_+^{\rm post}|\psi_+^{\rm pre}\rangle\langle\psi_-^{\rm pre}|\psi_-^{\rm post}\rangle). \tag{7}$$

* Note that this probability amplitude is conditioned by the system pre-selection in $|\Psi\rangle^{\rm pre}$.

This is asymptotic value of geometric phase just before the disappearance of interference, when both the numerator and denominator of the ratio in (5) have the same dependence on θ and differ only by the coefficients $|\mathcal{E}_1(0)|^2 - |\mathcal{E}_2(0)|^2$ and $|\mathcal{E}_1(0)|^2 + |\mathcal{E}_2(0)|^2$, respectively. In this case, expression (5) loses its dependence on θ.

3. Discussion

The considered scheme has a notable feature. In a usual experimental setup with initial preparation of $|\Psi\rangle^{\rm pre}$, neither p_1 nor p_2 depends on θ since the phase shift and the measurement occur in different arms of interferometer. Any such dependence would violate the causality principle. The probabilities p_i acquire a θ-dependence after specifying a subset of experimental outcomes corresponding to successful post-selection of $|\Psi\rangle^{\rm post}$. Similarly, there is no θ-dependence in the ensemble specified solely by post-selection, with a totally unknown input state. One can make a conclusion that interference and geometric phases exist in between the acts of pre- and post-selection and originate from them.

The crucial point in the considered interferometric scheme is the existence of two alternative directions of spatial motion which may form a superposition. If the internal state space of the system is two-dimensional, i.e. the system is a qubit, the addition of two spatial states $|0\rangle$ and $|1\rangle$ makes an effective two-qubit system. It is then logical to pose the next question: Is it possible to observe the interference in a real two-qubit system with pre- and post-selection without an interferometer?

Consider the scheme presented in Fig. 2, which is different from that in Fig. 1 despite some obvious similarities. While Fig. 1 depicts the spatial structure of a MZ interferometer, Fig. 2 represents the space–time scheme of preparation, evolution and post-selection of a two-qubit system. Triangular blocks in the style of the quantum diagrammatic technique proposed in [16] stand for pre- and post-selection acts. On the worldline of qubit 1 there is a region where it interacts with the environmental qubit, while qubit 2 undergoes selective phase transformation

$$|\sigma\rangle_2 \to \exp[\mathrm{i}\theta\sigma]\,|\sigma\rangle_2 \quad (\sigma = 0,1). \tag{8}$$

This selectivity compensates for a lack of an actual interferometer.

The states $|\Psi\rangle^{\rm pre}$ and $|\Psi\rangle^{\rm post}$ can be written in the form

$$|\Psi\rangle^{\rm pre,\,post} = \sum_{\sigma_1,\sigma_2} \psi^{\rm pre,\,post}(\sigma_1,\sigma_2)\,|\sigma_1\rangle_1 \otimes |\sigma_2\rangle_2\,, \tag{9}$$

Then the probability amplitude defined analogously to (4) is

$$A_i(\theta) = \sum_{\sigma_1,\sigma_1',\sigma_2} \psi^{\rm post\,*}(\sigma_1', \sigma_2)\psi^{\rm pre}(\sigma_1,\sigma_2) \times \exp[\mathrm{i}\theta\sigma_2]\,\langle \sigma_1' \,|\hat{\mathcal{E}}_i|\,\sigma_1\rangle, \tag{10}$$

where $\hat{\mathcal{E}}_i$ is the same as $\hat{\mathcal{E}}_i(1)$ from (3). It can be easily noted that the necessary condition for interference (and geometric phase) – θ-dependence in (5) – is the states $|\Psi\rangle^{\rm pre}$ and $|\Psi\rangle^{\rm post}$ being entangled. Indeed, if they are separable, i.e.

$$\psi^{\rm pre,\,post}(\sigma_1,\sigma_2) = \psi_1^{\rm pre,\,post}(\sigma_1)\,\psi_2^{\rm pre,\,post}(\sigma_2), \tag{11}$$

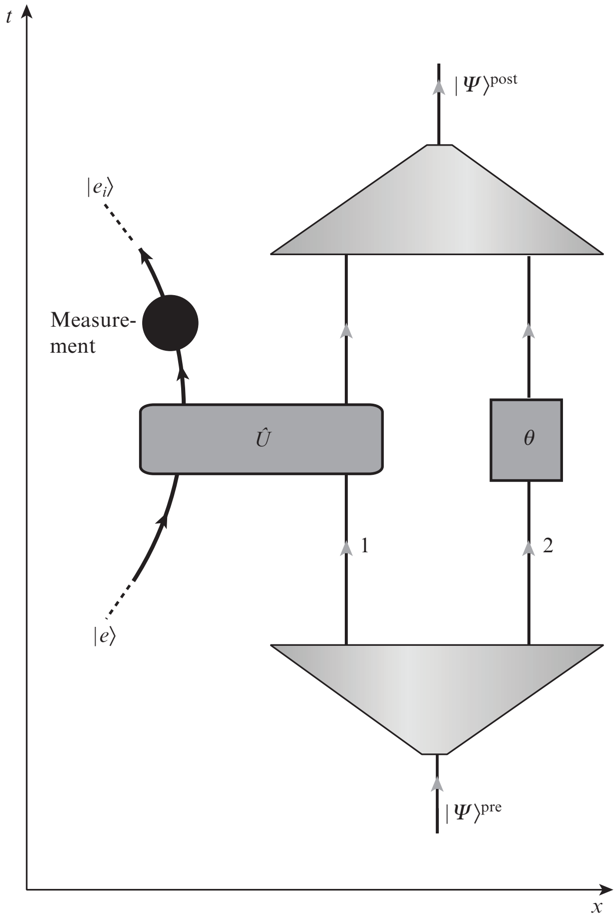

Figure 2. Space–time diagram of pre- and post-selection of states of a two-qubit system with an arrangement similar to Fig. 1 in between the selection acts.

the θ-dependence in $A_i(\theta)$ reduces to a phase factor:

$$\sum_{\sigma} \psi_2^{\rm post\,*}(\sigma)\,\psi_2^{\rm pre}(\sigma)\exp[\mathrm{i}\theta\sigma], \tag{12}$$

which does not depend on i and hence does not enter equation (5). Remarkably, for the effective two-component system considered above the states (2) do not need to be entangled to observe interference.

As a conclusion, we have demonstrated the existence of interference phenomena stipulated by pre- and post-selection procedures which are performed on the state of the quantum system. These phenomena reveal themselves in the statistics of system–environment interactions in the time interval between pre- and post-selection. The geometric phase introduced in the framework of the generalised operational approach accompanies the aforementioned interference phenomena.

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