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# Role of spontaneous emission in the formation of the steady-state optical spectrum of a diode laser

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***Abstract.*** **Based on Maxwell's equations, we derive a van der Pol-type equation for self-oscillations in the cavity of a diode laser. Its solution is self-oscillations (lasing) even in the absence of spontaneous emission. This solution differs fundamentally from solutions obtained using rate equations containing an additional term in the form of spontaneous emission intensity. Taking into account spontaneous emission in the van der Pol model makes calculation results more realistic and allows one to find parameters characterising diode laser output coherence, including the spectral densities of laser light amplitude and phase fluctuations.**

***Keywords:*** *diode lasers, spontaneous emission, fluctuations in parameters.*

## 1. Introduction

The question of what role spontaneous emission plays in the formation of a spectrum received a clear answer just a few years after the advent of lasers (see e.g. Refs [1 – 4]). In particular, for the diode laser a consistent theory taking into account spontaneous emission in single-frequency lasing mode was proposed in excellent reports by Vahala and Yariv [5, 6]. As shown in their and others' fundamental works, the presence of spontaneous emission sources in an inherently nonlinear self-oscillating laser system leads to intermixing of spontaneous and stimulated emissions. The result is amplitude and frequency modulations of lasing waves by a random signal. Thus, the role of spontaneous emission in a laser is to act as an inherent source of noise, which, in particular, determines the fine structure of the emission spectrum (the lower boundary of the laser linewidth), i.e. the laser coherence limit. Basically, this is what limits the effect of spontaneous emission on the formation of a steady-state optical spectrum.

Note that it is just such mode that we consider here and in what follows, excluding possible transient modes, for example, those taking place when pumping is 'abruptly' switched on or resulting from repeated suppression of lasing at an unstable operating point. In such modes, the role of spontaneous emission can be considerably more important than in the case of steady-state lasing.

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The question of the role of spontaneous emission in lasers might seem to be exhausted. Nevertheless, theoretical papers exist and continue to be published in which the diode laser is treated separately from other types of lasers. Such theoretical work can be exemplified by Refs [7, 8]. In these and other reports, use is made of an approach in which a laser field is considered essentially as amplified and spectrally filtered spontaneous emission. This physical meaning is suggested by theory proposed in Refs [7, 8], in which the spectral intensity distribution over modes can be represented as a fraction whose numerator is the spectral density of spontaneous emission and whose denominator is proportional to the difference between the loss and saturated gain. This difference characterises the incomplete compensation for the loss by the gain for each mode, because this mode in the cavity receives some additional constant energy in the form of spontaneous emission. In Suhara [9], this is reflected explicitly by formulas (6.40b) and (6.42). In Ivanov et al. [7], it follows unambiguously from the simplest equation (16). A similar equation (13), modified to take into account 'two-photon absorption', was used in Ref. [8]. Based on this approach and a simple solution to rate equations, it was concluded [7 – 9] that the experimentally observed multimode operation of a diode laser was a consequence of spontaneous emission along with the two-photon absorption mechanism. This conclusion has no physical meaning and is solely the result of an erroneous theoretical approach, which will be discussed below in greater detail.

This approach was earlier referred to as 'an asymptotic lasing threshold' by some authors. It is based on rate equations for some average intensity and inversion. Their relation to foundations of electrodynamics in the form of Maxwell's equations can only be seen, if any, in the form of the conservation-of-energy principle, and to a very limited extent. To these equations, some determinate functions, which are interpreted as the contribution of spontaneous emission, were added 'manually' (heuristically).

It is worth noting that misbeliefs about the role of 'foreign' signal sources in self-excited oscillators, which include lasers, have a long history. They were first dispelled at the beginning of the 20th century in radio engineering (see e.g. Pontryagin et al. [10], Bershtein [11] and Rytov [12, p. 414]) in analysis of Thomson's lamp oscillator. This occurred again after the advent of masers and lasers: the approach in question was criticised in the initial stages of the development of laser theory. In the field of diode lasers, however, these misbeliefs persisted for a much longer time, which is surprising because works providing sufficiently detailed insight into this issue [5, 6] were published long ago.

One possible cause of this erroneous approach is that a number of parameters characterising the diode laser as a self-oscillating system differ significantly from analogous parameters of other types of lasers. This refers primarily to parameters such as the spectral density of spontaneous emission 'in a mode' and the relation between the changes in the real and imaginary parts of permittivity in response to a change in inversion.

Of course, Refs [7, 8] and other, similar works could be ignored as erroneous, without dwelling on them, but the approach that uses rate equations containing an additional term related to spontaneous emission often 'roams' from one paper to another in the literature. The process continues at present, and this approach appears in a large number of books dealing with diode lasers (e.g. in Suhara [9]), so there is a need to separately analyse its incorrectness.

In accordance with the above, the purpose of this work is to demonstrate once more the effect of spontaneous emission on the output spectrum of a diode laser with allowance for distinctive features of its parameters, using a consistent analysis (in semiclassical theory) based on Maxwell's equations.

Since a considerable part of the physical content of this paper is more or less present in the extensive literature dealing with other types of lasers, this paper inevitably takes the form of a review. Moreover, this paper is aimed at representing the theory of the diode laser as a section of quantum radiophysics, which differs from its other sections only in quantitative characteristics of models, and not in their physical meaning and approaches.

For convenience of understanding the paper, we endeavoured to sequentially present approaches to the theory of the spectrum and intensity and inversion fluctuations of a diode laser in single-frequency mode in detail, beginning with foundations of electrodynamics.

In the original part of the paper, we derive expressions for the spontaneous emission factor. It plays a key role in the quantitative description of the effect of spontaneous emission on output laser beam parameters. From the methodological point of view, in this paper we use terms and parameters the most adequate to experiments with diode lasers. We endeavoured to make the final results as accessible for comparison with measurement data as possible.

## 2. Complex dielectric permittivity and polarisation of a semiconductor medium

To sequentially find an equation for the field amplitude, it is necessary to use Maxwell's equations for the medium of the optical cavity of a diode laser. These equations contain the electric induction vector $D$, which characterises only the medium and the interaction of an electromagnetic wave with it. It is the sum of the electric field vector $\mathcal{E}$ (averaged over a physically small volume) and polarisation vector $\mathcal{P}$ multiplied by $4\pi$. The polarisation vector $\mathcal{P}$ is in turn the average dipole moment per unit volume:

$$\bar{\mathcal{P}} = \frac{1}{v}\sum_j q_j \bar{r}_j = \frac{1}{v(r)}\sum_j^{r_j\in v} q_j \bar{r}_j^{\,\text{ext}} + \frac{1}{v(r)}\sum_j^{r_j\in v} q_j \bar{r}_j^{\,\text{int}} = \overline{\mathcal{P}}^{\text{ext}} + \overline{\mathcal{P}}^{\text{int}}, \tag{1}$$

where $\bar{r}_j(t) = \bar{r}_j^{\,\text{ext}}(t) + \bar{r}_j^{\,\text{int}}(t)$; $v$ is a physically small volume; $q_j$ is the charge of the $j$th particle; and $\bar{r}_j$ is its radius vector, which is the sum of two vectors: displacement $\bar{r}_j^{\,\text{ext}}$ under the effect of an external field, i.e. the response of an elementary charge to the external (stimulating) influence, and the vector $\bar{r}_j^{\,\text{int}}$ of intrinsic motion under the effect of internal forces existing in the medium independent of the external field.

Consider first sufficiently weak fields, in which the response of a charge to an external field arbitrarily varying with time $t$ can be taken to be linear. In such a case, without loss of generality, only one harmonic, of frequency $\omega$, can be considered as an external field, i.e. we take that

$$E(t) \sim \frac{1}{2}[A(\omega)\exp(-\mathrm{i}\omega t) + \text{c.c.}].$$

Here and in what follows, we use $E$ for real values and $\mathcal{E}$ for the corresponding complex values of the electric field intensity. The time dependence $r_j^{\text{ext}}(t)$ is also proportional to the field, with a proportionality coefficient $\alpha(\omega)$, i.e. $r_j^{\text{ext}}(t) = \alpha_j(\omega)A(\omega)\exp(-\mathrm{i}\omega t)$ as the response of a linear system to a harmonic force. The polarisation amplitude induced by an external field is then given by

$$P^{\text{ext}}(\omega) = \frac{1}{v}\sum_j q_j\alpha_j(\omega)A(\omega) = \chi(\omega)A(\omega), \tag{2}$$

where $\chi(\omega)$ is a so-called susceptibility, a complex quantity in the general case, characterising the medium in electrodynamics.

The complex dielectric permittivity $\varepsilon(\omega)$ is defined as a proportionality coefficient between a Fourier component of the external field-induced electric induction $D(\omega)$ and the field, as represented by the relations

$$D(\omega) = A(\omega) + 4\pi P^{\text{ext}}(\omega) = A(\omega)[1 + 4\pi\chi(\omega)] = \varepsilon(\omega)A(\omega), \tag{3}$$

where $\varepsilon(\omega) = 1 + 4\pi\chi(\omega)$.

Thus, two distinct types of charge motion are responsible for all electrodynamic characteristics of a medium. One of them is the response of charges to an external field, as represented by the function $\varepsilon(\omega)$, and the other is driven by internal forces, which in turn depend on the state of the system. The latter type of motion is represented by the function $\bar{r}_j^{\,\text{int}}(t)$ in (1). Mathematically, this is expressed by the formulation in which the general solution to a linear differential equation can be represented by the sum of solutions to the homogeneous equation and a partial inhomogeneous equation. The latter type of motion includes, for example, the thermal motion of charges, which can be viewed as some chaotic currents distributed at random in the medium and generating electromagnetic waves, and the motion of charges due to possible random external forces. This type also includes the motion of a charge accompanying a spontaneous transition of a radiating element from its upper energy state to its ground state. A general characteristic of this type of motion is its random nature.

Clearly, to find the response function $\alpha(\omega)$ of charges present in a medium, it is necessary to use quantum-mechanical calculations of the forced motion of a huge number of interacting charges in the form of electrons and ionic cores of a crystal. The problem turns out to be so complex that it cannot be solved exactly. Nevertheless, we can approximately formulate a physical model for a qualitative analysis of $\alpha(\omega)$ and, accordingly, $\varepsilon(\omega)$. The model considers a set of harmonic oscillators with $\varepsilon(\omega)$ represented in the following form:

$$\varepsilon(\omega) = 1 + 4\pi\sum_j\left(\frac{q_j^2}{m_j}\right)\frac{f_j}{\omega_j^2 - 2\mathrm{i}\tilde{\gamma}_j\omega - \omega^2}. \qquad (4)$$

The sum is taken over all $j$ oscillators in a physically small volume of a crystal, and the result is normalised to this volume. Each oscillator is determined by its charge $q_j$, its mass $m_j$, its resonance frequency $\omega_j$, its damping constant $\tilde{\gamma}_j$ and a dimensionless coefficient $f_j$, referred to as the oscillator strength (whose sign depends on the direction of the transition). These quantities can, in principle, be calculated in quantum theory.

In reality, however, this formula is of little utility for quantitatively determining the $\varepsilon(\omega)$ of a semiconductor medium with an accuracy necessary for practice. Nevertheless, it allows $\varepsilon(\omega)$ to be qualitatively divided into two distinct parts. The point is that not all oscillators contributing to $\varepsilon(\omega)$ are equivalent. Let us separate out in (4) the oscillators whose frequencies are far from resonance with optical radiation. They correspond to electron transitions to high energy states, which are responsible for absorption in the ultraviolet and X-ray spectral regions. Because of the large number of such oscillators, they determine the refractive index of the medium (the real part of its $\varepsilon$) and cause a small 'background' absorption. This part of the dielectric permittivity is denoted as $\tilde{\varepsilon}(r,\omega)$. The other part, denoted as $\varepsilon_a(r,\omega,N)$, is due to the oscillators with resonance frequencies near the external field frequency. The real and imaginary parts of this contribution are comparable (in magnitude) and depend on the difference between the populations in the conduction and valence bands, because $f_j$ changes sign in response to a change in the direction of the electron transition (from the valence band to the conduction band or in the opposite direction). With this definition, $\varepsilon_a(r,\omega,N)$ can be written in the following form:

$$\varepsilon_a(r,\omega,N) = \varepsilon_a(r,\omega,N_0) + \frac{\partial\varepsilon_a}{\partial N}(N - N_0), \qquad (5a)$$

$$\varepsilon(r,\omega,N) = \varepsilon_0 + \delta\varepsilon,$$

$$\varepsilon_0(r,\omega,N_0) = \tilde{\varepsilon}(r,\omega) + \varepsilon_a(r,\omega,N_0), \qquad (5b)$$

$$\delta\varepsilon(r,\omega,N) = \frac{\partial\varepsilon_a}{\partial N}(N - N_0).$$

The imaginary part of $\varepsilon(r,\omega,N)$ represents optical absorption (amplification) in the bulk of the medium, including that resulting from resonance transitions between the conduction and valence bands. It depends on the concentration of nonequilibrium carriers, $N(r)$ (inversion), in the active region of the laser cavity and differs from zero only for $r \in V_{\rm act}$, where $V_{\rm act}$ is the volume of the active region. This dependence is represented in (5) in linearised form with respect to $N$. In what follows, the value at which the medium has a transmission edge at a given frequency $\omega$ will be used as $N_0$.

The representation of $\varepsilon(r,\omega)$ in the form (5) is fairly typical of analysis of various types of lasers. Distinctions are possible only in interpretation of the terms. For example, in the case of solid-state lasers $\tilde{\varepsilon}(r,\omega)$ is taken to mean the dielectric permittivity of the laser host material and $\delta\varepsilon$ is the contribution of active ions.

Clearly, quantitative values of $\varepsilon_0(r,\omega)$ and $\delta\varepsilon(r,\omega,N)$ cannot be derived purely theoretically with required accuracy. Nevertheless, the use of independent experimental data for various semiconductor media and optical measurements in combination with analytical calculations make it possible in a number of cases to find functions approximating $\varepsilon_0(r,\omega)$ and $\delta\varepsilon(r,\omega,N)$ with acceptable accuracy.

One distinctive feature of the diode laser is that $\varepsilon_0(r,\omega)$ is a significant function of coordinates. Even though a diode is a single monocrystalline chip, it comprises a large number (up to hundreds or even more) of layers or regions differing in chemical composition and, hence, in dielectric permittivity. In the vast majority of cases, it is the spatial distribution of $\varepsilon_0(r,\omega)$ which determines the quantitative parameters of the dielectric cavity of a diode laser that are of great importance for simulation.

The characteristic size of the layers is comparable to or even much smaller than optical wavelengths. This means that, from the viewpoint of a single scale of electromagnetic waves, the formulation of and approach to problems related to the amplification and propagation of waves in a diode cavity correspond to problems pertaining to the microwave range, even though the frequency of such waves corresponds to the IR and optical regions. Note that, in the case of solid-state and gas lasers, analogous problems can be solved by methods characteristic of physical (and sometimes even geometrical) optics.

Another distinctive feature of the diode laser is that the derivative $\partial\varepsilon/\partial N$ comprises not only an imaginary part proportional to the cross section of a stimulated transition, $\sigma(\omega)$, but also an appreciable real part with a coefficient $R$, i.e.

$$\frac{\partial\varepsilon}{\partial N} = -\frac{\sigma(\omega)n_{\rm act}c}{\omega}(R + \mathrm{i}), \qquad (6)$$

where $n_{\rm act} = \sqrt{\varepsilon_{\rm act}}$ is the refractive index of the active region; $\varepsilon_{\rm act}$ is its dielectric permittivity; and $c$ is the speed of light. Physically, this means that a change in inversion in the active medium is accompanied by changes in not only the gain but also the refractive index of the medium. Qualitatively, this is an obvious result for all types of lasers if we take into account that a gain is produced in the spectral region of anomalous dispersion. However, the coefficient $R$ is rather small for most types of lasers and can be left out of account. The point is that the operating laser transition can be thought of as a transition in a two-level system. Since lasing typically takes place in the centre of a spectral line, where the contribution to the real part of susceptibility is zero because of the anomalous dispersion, $R$ is relatively small. In this case, $R$ differs from zero only because the laser transition differs from that in an ideal two-level system. For example, the $R$ of a Nd:YAG active medium ranges from 0.2 to 1.2 [13]. The situation is quite different in the case of a semiconducting active medium, where the laser transition occurs between two energy bands, each having its own quasi-continuous energy spectrum, so that the amplification–absorption spectral line is extremely asymmetric. Besides, the real part of susceptibility is contributed by free charge carriers, because the plasma frequency depends on their concentration. This contribution amounts to one-third of the contribution from anomalous dispersion. As a result, the coefficient $R$ of typical semiconductor media can fall in the range 2–6. This coefficient became widely known after work by Henry [14] in the context of the increase in diode laser linewidth because of the spontaneous emission. Since that time, it has often been referred to as the Henry factor. Note, to be fair, that this coefficient was introduced and used significantly earlier by other researchers (see e.g. Refs [2, 3]), but, because it is very small in the case of gas lasers and typical solid-state lasers, it was less needed than in the case of diode lasers.

To complete the definition of $\varepsilon(\omega)$ in the form (5), attention should be paid to another important circumstance, namely, that the response of a system of charges in a medium to an external field is linear, which was first formulated as a condition for defining $\varepsilon(\omega)$. For this reason, the expressions for $\varepsilon(\omega)$ do not include field strength in explicit form. On the other hand, it is well known that a self-oscillating system is inherently nonlinear, so it is impossible to obtain an adequate solution for it in the framework of only a linear approach. In this context, consider how a strong field can change the state of a medium so that it will lead to a change in its response function. Note first of all that $\varepsilon_0(\omega)$ is determined by oscillators far from resonance with light, so the possible nonlinear component of $\varepsilon_0(\omega)$ is rather small, because the scale is set by the ratio of the external field strength to the intraatomic field strength. Such nonlinear susceptibility is characteristic of most transparent media. In our analysis, it plays an insignificant role and will be omitted in what follows, even though the $\varepsilon_0(\omega)$ of typical semiconductors usually exceeds 10.

A quite different situation occurs for nonlinearity $\delta\varepsilon$, because it is related to the population of operating levels and the polarisation state. A proper analysis of polarisation dynamics for operating transitions is extremely complex. In some form, it can be done by analysing the dynamics of the density matrix and its diagonal and off-diagonal elements, for example, at least in a tau approximation. Even though there has been extensive theoretical work in this direction, it is difficult to expect that the calculational models used completely described the entire physics of phenomena that occur in the active region of an advanced diode laser. Indeed, typical free carrier concentrations are $10^{18}$ to a few times $10^{19}$ cm$^{-3}$. In intermetallic semiconductors, which are used as gain media of diode lasers, even above room temperature by 100 K and more such concentrations correspond to a degenerate Fermi gas with considerable Coulomb interaction between particles. Density matrix elements cannot be factorised in the form of the product of one-electron Bloch functions or by any other simple system of functions corresponding to 'pure' states.

Approximate calculations for an electron state due to Coulomb interaction give a characteristic 'energy broadening' above 10 meV, which corresponds to characteristic phase relaxation times of off-diagonal elements (transverse relaxation time) in the femtosecond range (under $10^{-13}$ s). This is confirmed to some extent by experimental data obtained in pump–probe measurements with femtosecond optical pulses [15]. Therefore, the dynamics of the electron system in each energy band can be viewed as the dynamics of a quasi-equilibrium Fermi gas characterised by only local concentration and temperature if characteristic times exceed $10^{-13}$ s.

Thus, it is obvious that, if the field amplitude in a laser cavity varies no faster than in $10^{-13}$ s, the polarisation of the medium 'has time to forget' its preceding states and quasi-statically keeps track of field dynamics. Its amplitude is proportional to the product of the field amplitude and the difference between the diagonal density matrix elements. The phase shift of the polarisation response with respect to the field amplitude is taken into account by the spectral form factor, so in the case of a laser system there is no need to additionally use an equation for polarisation. The complex susceptibility due to resonance transitions and, hence, $\delta\varepsilon$ quite adequately describe the dynamics of field–medium interaction on account of the dynamics of the diagonal density matrix elements or, with allowance for the above, the dynamics of the carrier concentration in the conduction and valence bands.

This significantly simplifies analysis because the nonlinearity of a laser system due to gain saturation is automatically implicitly taken into account in (5) for $\delta\varepsilon$ through the dependence of $N$ on the intensity of an electromagnetic wave. Below, this dependence will be found for a simplified model of an active region from the carrier balance equation.

## 3. Carrier balance in the active region of a laser

The typical geometry of the active region of a modern diode laser usually has the form of a flat layer of thickness $d$ (from a few nanometres to 100 μm), with carrier injection along the normal to it. The carrier balance equation has the form

$$\frac{\partial N(r,t)}{\partial t} = \frac{J(r)}{ed} - \frac{N(r,t)}{\tau} + \frac{\omega\,\mathrm{Im}(\delta\varepsilon)\overline{E}^2(r,t)}{4\pi\hbar\omega} - \mathcal{D}\Delta N(r,t). \tag{7}$$

Here $J$ is the current density distribution; $e$ is the electron charge; $\tau$ is the spontaneous recombination time; $\overline{E}^2$ is the period-averaged square of the electric field intensity; and $\hbar\omega$ is the energy of an electron transition from the conduction band to the valence band. Ambipolar electron diffusion processes are taken into account by the last term in (7), which is proportional to the coefficient $\mathcal{D}$. Since the diffusion length usually far exceeds the thickness of the active layer, electron diffusion in the $x$ direction, normal to the layer, results in a completely flat concentration profile. Therefore, diffusion in the $x$ direction can be left out of consideration and the concentration can be taken with high accuracy to be independent of this coordinate. As to the other two directions, along the layer ($y$ and $z$), the situation is somewhat more complex. In this case, one can solve the problem in the form of a 2D equation (7) (1D equation for diodes emitting along the layer $d$), thus finding the spatial carrier distribution. In what follows, the steady-state spatial electron distribution $N(r)$ is thought to be found by solving Eqn (7) and to have the form

$$N(r) = \mathcal{N} f(r),$$

where the dimensionless function $f(r)$ is determined by the laser configuration and varies little with pump current $J$ by virtue of a number of circumstances characteristic of the diode lasers under consideration. Because of this, in what follows $f(r)$ is thought to retain its form upon changes in pump current, in particular above threshold, and to be normalised to unity at its maximum, which is located in the origin: $f(0) = 1$. Note that this is a very stringent requirement (assumption), which is extremely rarely met in practice. It imposes a constraint on the spatial pump distribution $J(r)$ and the spatial $\bar{E}^2$ distribution. This is quite consistent with the well-known fact that diode lasers, e.g. those with a Fabry–Perot cavity, extremely rarely operate in single-frequency mode even just above threshold.

Let us represent the field amplitude in factorised form:

$$E(r,t) = \frac{1}{2}[\mathcal{E}(t)\bar{u}(r) + \text{c.c.}], \tag{8}$$

where $\mathcal{E}(t) = A(t)\exp(-\mathrm{i}\omega t)$; $\bar{u}(r)$ is a complex dimensionless vector function normalised to unity magnitude in the origin, i.e. $\langle\bar{u}(0)\bar{u}^*(0)\rangle = 1$; and $A(t)$ is a 'slow' field amplitude.

In a quasi-steady state, where $A(t)$ varies considerably more slowly than $\tau^{-1}$, the derivative $\partial N/\partial t$ in (7) can be neglected and, taking into account (5) and (6), we obtain

$$N(t,r) = \mathcal{N}_{\text{tr}} f(r)\left(1 + \frac{p}{1 + |\mathcal{E}|^2/E_{\text{s}}^2}\right). \tag{9}$$

Here

$$p = \frac{J - J_{\text{tr}}}{J_{\text{tr}}};\ J_{\text{tr}} = \frac{e\mathcal{N}_{\text{tr}}V_{\text{act}}}{\tau};$$

$$\mathcal{N}_{\text{tr}} = \frac{N_0 V_{\text{m}}}{V_{\text{mc}}};\ E_{\text{s}}^2 = \frac{8\pi\hbar\omega V_{\text{act}}}{\tau\sigma n_{\text{act}} c V_{\text{mc}}};$$

$$V_{\text{mc}} = \int\langle\bar{u}(r)\bar{u}^*(r)\rangle f(r)\,\mathrm{d}V;\ V_{\text{m}} = \int\langle\bar{u}(r)\bar{u}^*(r)\rangle\,\mathrm{d}V;$$

$$V_{\text{act}} = \int f(r)\,\mathrm{d}V.$$

These designations result from the averaging of the main quantities over the active region by integrating over the space of the balance equation (7). They have the following physical meaning: $p$ is the relative difference between the pump current $J$ and the transparency current $J_{\text{tr}}$; $\mathcal{N}_{\text{tr}}$ is the average transparency concentration; the quantity $E_{\text{s}}^2$ is proportional to saturation intensity; and $V_{\text{act}}$, $V_{\text{m}}$ and $V_{\text{mc}}$ are the effective active region, mode and mode–carrier overlap volumes. With the quantities introduced above, Eqn (5b) for $\delta\varepsilon$ can be written as

$$\delta\varepsilon \approx -\frac{cn_{\text{act}}}{\omega}G(R + \mathrm{i})\left(\frac{f(r)p}{1 + |\mathcal{E}|^2/E_{\text{s}}^2}\right), \tag{10}$$

where $G = V_{\text{m}}\sigma(\omega)N_0/V_{\text{mc}}$ is the unsaturated material gain coefficient (in inverse centimetres) in the active region. In (10) we use the fact that the averaged expression

$$\int\langle\bar{u}(r)\bar{u}^*(r)\rangle\,[N_{\text{tr}}(r) - N_0]\,\mathrm{d}V$$

is zero according to the definition of $N_{\text{tr}}(r)$. Now we have all quantities needed to write down the equation for the mode amplitude.

## 4. Equation for the cavity mode amplitude in a diode laser without spontaneous emission sources

For a static field of the form (8) to meet Maxwell's equations in a cavity, $\bar{u}(r)$ should meet the equation

$$\text{rot}\,\text{rot}(\bar{u}_k(r)) = \frac{\tilde{\omega}_k^2}{c^2}\varepsilon_0(r,\omega_k,N_0)\bar{u}_k(r), \tag{11}$$

where $\tilde{\omega}_k = \omega_k - \mathrm{i}\gamma_k$, and boundary conditions on a surface $S$ that encompasses the cavity. This surface can be fictitiously divided into two parts. In one of them ($S_0$), the electric and magnetic fields are zero. Through the other ($S$), the cavity is connected to the ambient medium. Equation (11) defines a set of $\bar{u}_k(r)$ eigenmodes and $\tilde{\omega}_k$ complex eigenfrequencies of the dielectric cavity. Thus, the cavity is fully defined by setting $\varepsilon(r,\omega,N)$ for a particular type of laser diode. For example, this can be edge-emitting and vertical cavity surface-emitting laser diodes. Clearly, a solution to Eqn (11) can only be found for a particular form of $\varepsilon(r,\omega,N)$. Let $\bar{u}_k(r)$ and $\tilde{\omega}_k$ be already found by solving Eqn (11) as a separate electrodynamic problem. It should be noted that a nontrivial solution to (11) exists not for all $\varepsilon(r,\omega,N)$ functions, so to analyse characteristics of such cavities it is reasonable to use an approach described by Vainshtein [16] in which an 'equivalent' cavity is introduced into analysis. The approach builds on the fact that, if frequency $\omega$ is brought to the complex plane and the corresponding $\gamma$ is found, the real cavity can be replaced by an equivalent cavity having zero fields on the entire outer surface $S$.

In what follows, we assume that the eigenfunction $\bar{u}_k(r)$ thus found varies little with concentration $N$. Thus, Eqn (11) can be thought of as an invariant for a particular mode. Consider first the field of one mode whose frequency $\omega_c$ is close to the spectral maximum in gain $G(\omega)$. For this mode, Maxwell's equations lead to the equation

$$\frac{\partial^2 D(r,t)}{\partial t^2} + (\omega_c - \mathrm{i}\gamma_c)^2\varepsilon_0(r,\omega_c,N_0)\mathcal{E}(t)\bar{u}_c(r) = 0, \tag{12}$$

where $\omega_c$ is the resonance frequency of the cavity and $\gamma_c$ is the cavity field attenuation constant, which depends on both the optical loss in the bulk of the cavity and the output coupling losses. Further, we represent $\varepsilon(r,\omega,N)$ as a function of frequency in linearised form in the vicinity of frequency $\omega_c$:

$$\varepsilon(r,\omega) = \varepsilon_0(r,\omega_c,N_0) + \delta\varepsilon + \frac{\partial\varepsilon_0}{\partial\omega}(\omega - \omega_c). \tag{13}$$

Carrying out standard operations for electric induction and taking into account (10) and (13), we obtain

$$\frac{\partial^2 D(r,t)}{\partial t^2} = \left\{\varepsilon_0\frac{\partial^2\mathcal{E}}{\partial t^2} - \omega_c\left[p\frac{f(r)cn_{\text{act}}G(1 - \mathrm{i}R)}{\omega_c(1 + |\mathcal{E}(t)|^2/E_{\text{s}}^2)} + i\Delta\varepsilon\right]\frac{\partial\mathcal{E}}{\partial t}\right.$$
$$\left. + \Delta\varepsilon\omega_c^2\mathcal{E}(t)\right\}\bar{u}_c(r). \tag{14}$$

Here, we use the relation

$$\frac{\partial^2}{\partial t^2}\int\frac{\partial\varepsilon}{\partial\omega}(\omega - \omega_c)\mathcal{E}(\omega)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega \approx$$

$$\approx -\mathrm{i}\omega_c^2 \frac{\partial\varepsilon}{\partial\omega}\left[\frac{\partial\mathcal{E}(t)}{\partial t} + \mathrm{i}\omega_c\mathcal{E}(t)\right],$$

and $\Delta\varepsilon = \omega_c(\partial\varepsilon/\partial\omega)$ determines dispersion. Note that induction $D(r,t)$ is defined according to (3), so that polarisation has no terms related to intrinsic charge motion $[\sim q_j r_j^{\mathrm{int}}(t)]$. Further, substituting (14) into (12), taking the scalar product with $\bar{u}_c(r)$ and integrating over the cavity, we obtain

$$\frac{\partial^2\mathcal{E}}{\partial t^2} + \left\{2\gamma_c\left[1 - p\frac{\tilde{\gamma}(1-\mathrm{i}\tilde{R})}{\gamma_c(1+|\mathcal{E}(t)|^2/E_{\mathrm{s}}^2)}\right] - \mathrm{i}\omega_c\Delta\tilde{\varepsilon}\right\}\frac{\partial\mathcal{E}}{\partial t} +$$

$$+\omega_c^2(1+\Delta\tilde{\varepsilon})\mathcal{E}(t) = 0, \qquad (15)$$

where the dimensionless parameter $\Delta\tilde{\varepsilon}$ characterises the total dispersion in the cavity and $\tilde{\gamma}$ is the effective (average) gain in the cavity. If $\bar{u}_c(r)$ and $\gamma_c$ are found by solving Eqn (11), the above parameters can be found as

$$\tilde{\gamma} = \frac{cn_{\mathrm{act}}G\tilde{\Gamma}}{2},$$

$$\tilde{\Gamma} = \mathrm{Re}\left[\frac{1}{U}\int_V (1-\mathrm{i}R)f(r)\langle\bar{u}_c\bar{u}_c\rangle \mathrm{d}V\right],$$

$$U = \int_V \varepsilon_0(\omega_c, N_0, r)\langle\bar{u}_c(r)\bar{u}_c(r)\rangle\,\mathrm{d}V,$$

$$\Delta\tilde{\varepsilon} \approx \left|\frac{1}{U}\int\Delta\varepsilon(r)\langle\bar{u}_c\bar{u}_c\rangle\,\mathrm{d}V\right|,$$

$$\tilde{R} = -\,\mathrm{Im}\left[\frac{1}{U\tilde{\Gamma}}\int_V (1-\mathrm{i}R)f(r)\langle\bar{u}_c\bar{u}_c\rangle \mathrm{d}V\right].$$

Here dimensionless $\tilde{\Gamma}$ and $\tilde{R}$ are analogues of the optical confinement factor and amplitude–phase coupling constant with allowance for the fact that $U$, the norm of the eigenfunction $\bar{u}_c(r)$, is complex-valued. The angle brackets denote the scalar product of the vectors.

Equation (15) is an equation for a harmonic nonlinear oscillator with a small $\tilde{\gamma}$ parameter. It is one of the van der Pol-type equations for self-oscillating systems, well studied in the theory of oscillations. The small parameter $\tilde{\gamma}$ is present as a coefficient of the nonlinear term $\sim(1+|\mathcal{E}|^2/E_{\mathrm{s}}^2)^{-1}$, whose physical meaning here is gain saturation as the average square of the oscillation amplitude, $|\mathcal{E}|^2$, rises.

Methods for a general analysis of such equations were developed as early as the first half of the 20th century. For this purpose, use is made e.g. of the van der Pol method or the Poincare method (see e.g. Andronov et al. [17]). In view of this, here we limit ourselves to known results. In the case under consideration, all solutions are characterised by two limit cycles. At a pumping level $p < p_{\mathrm{th}} = \gamma_c/\tilde{\gamma}$, there is one stable limit cycle, which degenerates into the $\mathcal{E}\equiv 0$ point. For $p > p_{\mathrm{th}}$, this point becomes an unstable operating point, whereas a circumference of radius $A_0 = E_{\mathrm{s}}\sqrt{2(p-p_{\mathrm{th}})/p_{\mathrm{th}}}$ in the van der Pol plane is a stable limit cycle. Thus, the pumping level $p = p_{\mathrm{th}}$ is the lasing threshold. It follows from Eqn (15) that the limit cycle parameters $|\mathcal{E}| = A_0$ and $\omega_0$ meet the relations

$$p_{\mathrm{th}} = \gamma_c/\tilde{\gamma},$$

$$\omega_0^2 - \omega_0(2\gamma_c\tilde{R} - \Delta\tilde{\varepsilon}\omega_c) - \omega_c^2(1+\Delta\tilde{\varepsilon}) = 0, \qquad (16)$$

$$2\gamma_c - 2\tilde{\gamma}p\frac{1-\mathrm{i}\tilde{R}}{1+|\mathcal{E}|^2/E_{\mathrm{s}}^2} = 2\mathrm{i}\gamma_c\tilde{R}.$$

They reflect the facts that the saturated gain above threshold, $2\tilde{\gamma}p(1+|\mathcal{E}|^2/E_{\mathrm{s}}^2)^{-1}$, is exactly equal to the loss $2\gamma_c$ and that the frequency $\omega_0 = \omega_c + \Delta\omega$ differs from the resonance frequency $\omega_c$ of a 'cold' cavity by $\Delta\omega \approx \gamma_c\tilde{R}(1+\Delta\tilde{\varepsilon}/2)^{-1}$ because of the carrier-induced change in the refractive index of the medium of the cavity at the lasing threshold. Gain saturation above threshold automatically means that the electron concentration is constant at the threshold level $\mathcal{N}_{\mathrm{th}} = \mathcal{N}_{\mathrm{tr}}(p_{\mathrm{th}}+1)$ and that the threshold pump current is $J_{\mathrm{th}} = e\mathcal{N}_{\mathrm{th}}V_{\mathrm{act}}/\tau$.

Thus, independent of the initial conditions, $|\mathcal{E}| > A_0$ or $|\mathcal{E}| < A_0$, all possible solutions to Eqn (15) tend to the form

$$\mathcal{E}(t) = A_0\exp(-\mathrm{i}\omega_0 t + \varphi), \qquad (17)$$

where $A_0 = E_{\mathrm{s}}\sqrt{2(p-p_{\mathrm{th}})/p_{\mathrm{th}}}$ and $\omega_0 \approx \omega_c + \gamma_c\tilde{R}(1-\Delta\tilde{\varepsilon}/2)$. This form corresponds to the generation of an ideal monochromatic oscillation with an amplitude $A_0$ and, accordingly, an output power $\sim A_0^2$, which is a linear function of the pumping level $p$ above threshold.

One important feature of the obtained solution $\mathcal{E}(t) = A_0\exp(-\mathrm{i}\omega_0 t + \varphi)$ is that it does not require or contain any external field sources, e.g. such as produce spontaneous emission. This fundamentally distinguishes (17) from solutions presented in Refs [7–9] and other works based on the 'asymptotic lasing threshold' approach.

## 5. Equation for the cavity mode amplitude in a diode laser with allowance for spontaneous emission sources

In what follows, an additional term in polarisation, due to intrinsic charge motion unrelated to a stimulating field $E(\omega)$, is included in $D(t)$. This term was ignored in (14). Equation (12) then takes the following form:

$$\frac{\partial^2\bar{D}(r,t)}{\partial t^2} + (\omega_c - \mathrm{i}\gamma_c)^2\varepsilon_0(r,\omega_c,N_0)\mathcal{E}(t)\bar{u}_c(r) = -4\pi\frac{\partial^2\bar{\mathcal{P}}^{\mathrm{int}}}{\partial t^2}. \qquad (18)$$

Let $\bar{\mathcal{P}}^{\mathrm{int}}$ be represented in the form

$$\bar{\mathcal{P}}^{\mathrm{int}}(r) = \frac{1}{v(r)}\sum_j^{r_j\in v} q_j\bar{r}_j^{\,\mathrm{int}}(t) = \frac{1}{v(r)}\sum_j^{r_j\in v}\bar{\boldsymbol{d}}_j(t),$$

where $\bar{\boldsymbol{d}}_j(t) = \boldsymbol{e}_j d_j(t)\exp(-\mathrm{i}\omega_0 t)$.

The right-hand side of Eqn (18) now contains a 'random' force vector (Langevin source). It results from an accelerated charge motion in the form of quasi-periodic, uncorrelated oscillations of dipoles in a physically small volume $v(r)$ centred at point $r$. Each dipole is described by its own random function $d_j(t)$ and unit vector $\boldsymbol{e}_j$. Clearly, the key lasing characteristics, such as the field amplitude $\mathcal{E}(t)$, concentration $N(t)$ and phase $\varphi(t)$ are then also random functions of time.

There are various methods of solving problems containing random quantities. Here we use one of them, based on stochastic equations with linearisation of nonlinear terms in the vicinity of their average 'operating' values. Previously [18], it was used to study the fine structure of the optical spectrum of a single-frequency diode laser at various fluctuations in cavity parameters, including the temperature and density of the medium. This method is valid if amplitude fluctuations are small compared to the corresponding quantities in the absence of fluctuation sources. In our case, this means that the pump current is well above threshold. In further analysis, we take into account this constraint.

Using the solutions (17) found above for the homogeneous equation (15), we represent the field amplitude $\mathcal{E}(t)$ and dielectric permittivity $\varepsilon$ for the inhomogeneous equation (18) in the following form:

$$\mathcal{E}(t) = A(t)\exp(-\mathrm{i}\omega_0 t),$$

$$A(t) = A_0[1 + a(t)]\exp[-\mathrm{i}\varphi(t)], \tag{19}$$

$$N(r,t) = N_{\mathrm{th}}(r)[1 + \tilde{n}(t)],$$

$$\varepsilon(r,\omega,N) = \varepsilon(r,\omega,N_{\mathrm{th}}) + \delta\tilde{\varepsilon}, \tag{20}$$

where

$$\delta\tilde{\varepsilon} = -\frac{cn_{\mathrm{act}}\sigma\mathcal{N}_{\mathrm{th}}}{\omega}(R + \mathrm{i})f(r)\tilde{n}(t);$$

$a(t)$ and $\varphi(t)$ are real functions describing field amplitude and phase fluctuations; and the dimensionless real function $\tilde{n}(t)$ describes electron concentration fluctuations. All these fluctuations are a dynamic response of the laser system to the random force represented by the right-hand part of (18).

Substituting the expressions (19) and (20) for $\mathcal{E}(t)$ and $\varepsilon(r,\omega,N)$ into (18) and then performing operations analogous to those in Section 4, namely, dot multiplication by $\bar{u}_c(r)$ and integration with respect to volume, we obtain an abbreviated equation for the amplitude $A(t)$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} - \Omega_0\tilde{n}(1 - \mathrm{i}\tilde{R})A = \frac{2\pi\mathrm{i}\omega_0}{U(1 + \Delta\tilde{\varepsilon}/2)}\sum_j^{r_j\in V_{\mathrm{act}}} d_j(t)\,\langle \boldsymbol{e}_j\bar{u}_c(r)\rangle, \tag{21}$$

where the frequency $\Omega_0 = \tilde{\gamma}/(1 + \Delta\tilde{\varepsilon}/2)$ characterises the inverse intracavity field build-up time. In deriving (21), we used the following relation stemming from (11):

$$(\omega_c - \mathrm{i}\gamma_c)^2\int\langle\bar{u}_c\bar{u}_c\rangle\varepsilon(\omega_c N_{\mathrm{tr}},,r)\,\mathrm{d}V = \omega_0^2\int\langle\bar{u}_c\bar{u}_c\rangle\varepsilon(\omega_0,N_{\mathrm{th}},r)\,\mathrm{d}V.$$

Substituting the expressions for the field amplitude (19) into (7) and (21), we obtain a system of equations for amplitude fluctuations $a(t)$, relative electron concentration fluctuations $\tilde{n}(t)$ and the 'instantaneous' frequency deviation $v = \mathrm{d}\varphi/\mathrm{d}t$:

$$\frac{\mathrm{d}a}{\mathrm{d}t} - \Omega_0\tilde{n} = F(t),\;\; F(t) = \mathrm{i}\xi\sum_j^{r_j\in V_{\mathrm{act}}} d_j(t)\,\langle \boldsymbol{e}_j\bar{u}_c(r)\rangle + \mathrm{c.c.},$$

$$\Omega_0 R\tilde{n} - v = \Phi(t),\;\; \Phi(t) = \xi\sum_j^{r_j\in V_{\mathrm{act}}} d_j(t)\,\langle \boldsymbol{e}_j\bar{u}_c(r)\rangle + \mathrm{c.c.}, \tag{22}$$

$$\tau\frac{\mathrm{d}\tilde{n}}{\mathrm{d}t} + \tilde{n}(1 + \theta\eta) + 2a\eta = h(t),\;\; \theta = \frac{\mathcal{N}_{\mathrm{th}}}{\mathcal{N}_{\mathrm{th}} - \mathcal{N}_{\mathrm{tr}}} = \frac{1 + p_{\mathrm{th}}}{p_{\mathrm{th}}},$$

$$\xi \approx \frac{\pi\omega_0}{A_0 U(1 + \Delta\tilde{\varepsilon}/2)}.$$

The dimensionless parameter $\theta$ characterises the difference between the transparency threshold of the active medium and the lasing threshold.

The equation for electron concentration fluctuations contains an additional Langevin source of possible fluctuations, $h(t)$, corresponding to pump current fluctuations $J = \bar{J} + \delta J(t)$, where $\bar{J}$ is the average current), e.g. due to shot noise:

$$\delta J(t) = J_{\mathrm{th}}h(t).$$

The system of equations (22) can readily be solved by Fourier expanding the functions $a(t)$, $\tilde{n}(t)$ and $v(t)$ and representing the result through the $F(\Omega)$, $\Phi(\Omega)$ and $h(\Omega)$ Fourier transforms of the functions $F(t)$, $\Phi(t)$ and $h(t)$:

$$F(t) = \int_{-\Delta}^{\Delta} F(\Omega)\exp(-\mathrm{i}\Omega t)\,\mathrm{d}\Omega,$$

$$\Phi(t) = \int_{-\Delta}^{\Delta} \Phi(\Omega)\exp(-\mathrm{i}\Omega t)\,\mathrm{d}\Omega,$$

$$h(t) = \int h(\Omega)\exp(-\mathrm{i}\Omega t)\,\mathrm{d}\Omega,\;\; \Omega = \omega - \omega_0.$$

The integration limits ($\Delta$) are such as to cover a physically significant frequency range. In particular, for $F(\Omega)$ and $\Phi(\Omega)$ we have $\Delta \geqslant \Delta\omega_{\mathrm{sp}}$, where $\Delta\omega_{\mathrm{sp}}$ is the spontaneous emission linewidth. In subsequent analysis, we use a 'slow' frequency, $\Omega$, along with $\omega_0$.

Slow dynamics of an individual dipole, $d_j(t)$, can also be represented in the form of a Fourier integral:

$$d_j(t) = \int_{-\Delta}^{\Delta} d_j(\omega_0 + \Omega)\exp(-\mathrm{i}\Omega t)\,\mathrm{d}\Omega.$$

For $F(\Omega)$ and $\Phi(\Omega)$ we then have

$$F(\Omega,r) = \mathrm{i}\xi\sum_j^{r_j\in V_{\mathrm{act}}} d_j(\omega_0 + \Omega)\,\langle \boldsymbol{e}_j\bar{u}_c(r)\rangle$$

$$- \mathrm{i}\xi^*\sum_j^{r_j\in V_{\mathrm{act}}} d_j^*(\omega_0 - \Omega)\,\langle \boldsymbol{e}_j\bar{u}_c^*(r)\rangle,$$

$$\Phi(\Omega,r) = \xi\sum_j^{r_j\in V_{\mathrm{act}}} d_j(\omega_0 + \Omega)\,\langle \boldsymbol{e}_j\bar{u}_c(r)\rangle$$

$$+ \xi^*\sum_j^{r_j\in V_{\mathrm{act}}} d_j^*(\omega_0 - \Omega)\,\langle \boldsymbol{e}_j\bar{u}_c^*(r)\rangle,$$

Solving the linearised system (22), we find the spectral densities of field amplitude, concentration and frequency fluctuations: $g_a(\Omega)$, $g_n(\Omega)$ and $g_v(\Omega)$, respectively. They are related to the corresponding amplitudes in the following way, standard for steady-state processes:

$$\overline{a(\Omega)a^*(\Omega')} = g_a(\Omega)\delta(\Omega - \Omega'),$$

$$\overline{\tilde{n}(\Omega)\tilde{n}^*(\Omega')} = g_n(\Omega)\delta(\Omega-\Omega'), \qquad (23)$$

$$\overline{v(\Omega)v^*(\Omega')} = g_v(\Omega)\delta(\Omega-\Omega').$$

In finding the spectral density of dipole oscillations, $d_j(t)$, we assume them to be statistically independent, so the following equalities are valid:

$$\overline{d_j(\omega)d_i(\omega')} = g_d(\omega)\delta_{j,i}\delta(\omega-\omega'),$$

$$\overline{|d_j(t)|^2} = \frac{d_0^2}{2} = \int_{-\Delta}^{\Delta} g_d(\omega_0+\Omega)\mathrm{d}\Omega. \qquad (24)$$

Here $\delta_{j,i}$ is the Kronecker delta and $d_0$ is the dipole oscillation amplitude. Finally, all key characteristics of fluctuations can be expressed through the spectral density of spontaneous emission sources, $g_d(\omega)$, which in turn can be found if we know $d_0^2$ and $\rho(\omega)$ (form factor of spontaneous emission lines). The amplitude $d_0$ can be found via direct calculation of the corresponding matrix element, but a simpler and more convenient approach is to employ the classical–quantum correspondence principle for dipole emission power. In classical theory, the power is $P = \omega_0^4 n_{\rm act} d_0^2/(3c^3)$; in quantum theory, we have $P = \hbar\omega_0/\tau$. Therefore, $d_0^2 = 3c^3\hbar(\omega_0^3 n_{\rm act}\tau)^{-1}$.

As to the form factor of spontaneous emission lines, representing it by the dimensionless function $\rho(\omega) \equiv \rho(\omega_0 + \Omega)$ normalised to unity at its spectral maximum, we obtain

$$g_d(\omega_0+\Omega) = \rho(\omega_0+\Omega)\frac{3c^3\hbar}{2\omega_0^3 n_{\rm act}\tau\Delta\omega_{\rm sp}}, \qquad (25)$$

where

$$\Delta\omega_{\rm sp} = \int_{-\Delta}^{\Delta}\rho(\omega_0+\Omega)\mathrm{d}\Omega.$$

Thus, using the expression for $g_d(\omega_0 + \Omega)$ and solutions to system (22) for fluctuation amplitudes, we finally obtain the following expressions for the spectral densities of these fluctuations:

$$g_a(\Omega) = \beta N\frac{\hbar\omega_0}{2A_0^2|U|(1+\Delta\tilde{\varepsilon}/2)\tau}[Q_a(\Omega)+Q_a(-\Omega)] + H_a(\Omega), \qquad (26a)$$

$$g_n(\Omega) = \beta N\frac{2\hbar\omega_0\eta^2[\rho(\omega_0+\Omega)+\rho(\omega_0-\Omega)]}{A_0^2|U|(1+\Delta\tilde{\varepsilon}/2)\tau^3[(\Omega_1^2-\Omega^2)^2+(\Omega\gamma)^2]} + H_n(\Omega), \qquad (26b)$$

$$g_v(\Omega) = \beta N\frac{\hbar\omega_0[Q_v(\Omega)+Q_v(-\Omega)]}{2A_0^2|U|(1+\Delta\tilde{\varepsilon}/2)\tau} + H_v(\Omega). \qquad (26c)$$

Here

$$N = \mathcal{N}_{\rm th}V_{\rm act};\quad Q_a(\Omega) = \frac{(\Omega^2+\gamma^2)\rho(\omega_0+\Omega)}{(\Omega_1^2-\Omega^2)^2+(\Omega\gamma)^2};$$

$$H_a(\Omega) = \frac{(\Omega_0 A_0)^2}{\tau^2[(\Omega_1^2-\Omega^2)^2+(\Omega\gamma)^2]}g_h(\Omega);$$

$$H_n(\Omega) = \frac{\Omega^2 g_h(\Omega)}{\tau^2[(\Omega_1^2-\Omega^2)^2+(\Omega\gamma)^2]};\ \Omega_1^2 = \frac{2\eta\Omega_0}{\tau};\ \eta = \frac{J-J_{\rm th}}{J_{\rm th}};$$

$$Q_v(\Omega) = \rho(\omega_0+\Omega)\frac{(\Omega_1^2-\Omega^2)^2+(\Omega_1^2\tilde{R}-\Omega\gamma)^2}{(\Omega_1^2-\Omega^2)^2+(\Omega\gamma)^2};$$

$$H_v(\Omega) = \frac{(\Omega_0\Omega\tilde{R})^2}{\tau^2[(\Omega_1^2-\Omega^2)^2+(\Omega\gamma)^2]}g_h(\Omega);\quad \gamma = \frac{1+\theta\eta}{\tau};$$

$$\beta = \frac{3\pi^2c^3\int f(r)\,\langle\bar{u}^*(r)\bar{u}(r)\rangle\,\zeta(r)\mathrm{d}V}{|U|\,(1+\Delta\tilde{\varepsilon}/2)n_{\rm act}\omega_0^2\Delta\omega_{\rm sp}\int f(r)\mathrm{d}V}$$

is a dimensionless coefficient; and

$$\zeta(r) = \frac{\overline{\langle \boldsymbol{e}_j\bar{u}_c^*(r)\rangle\langle \boldsymbol{e}_j\bar{u}_c(r)\rangle}}{\langle\bar{u}_c^*\bar{u}_c\rangle}.$$

The spectral density of current fluctuations, $g_h(\Omega)$, is given by

$$\overline{h(\Omega)h^*(\Omega')} = g_h(\Omega)\delta(\Omega-\Omega').$$

The parameter $\zeta(r)$ takes into account possible dipole moment anisotropy (for example, due to the quantum size effect) in the active region. In the isotropic case, $\zeta$ =1/33. The parameter $\beta$ can be interpreted as characterising the fraction of spontaneous emission in the laser cavity mode. Note that, in the limit of a homogeneous and isotropic medium with characteristic geometric cavity dimensions far exceeding optical wavelengths, this parameter tends to the inverse of the phase volume occupied by the electromagnetic wave field, i.e. to the frequently used expression in the form $\Delta^3kV/(2\pi)^3$ (the number of modes in this volume). If cavity dimensions are comparable to optical wavelengths, the definition of this factor is less simple and less general. In our calculations, it appears automatically as the electrodynamic problem is solved. Above, $\beta$ always appears together with the factor $N$, which denotes the total number of electrons in the active region. This corresponds to their physical meaning with respect to the effect of spontaneous emission in the form of expressions (26).

Thus, expressions (26) for the spectral density of amplitude, electron concentration and frequency fluctuations allow diode laser noise related to spontaneous emission and steady-state random pump current fluctuations to be characterised in sufficient detail.

Figure 1 shows spectral densities of fluctuations calculated using formulas (26) for a vertical cavity diode laser as an example. The parameters used in the calculation were similar to those obtained by Vas'kovskaya et al. [19] and Blokhin et al. [20] for vertical cavity surface-emitting lasers. Particular values are as follows:

$$\beta = \frac{\pi^2\mu K}{V_p n n^* n_{\rm act}}\Big(\frac{c}{\omega_0}\Big)^3 = 5\times10^{-4};$$

$$n = 3.5;\ n^* = 3.8;$$

$$n_{\rm act} = 3.6;\ \omega_0 = 2\pi c/\lambda;\ \lambda = 0.85\ \mu\mathrm{m};$$

effective cavity volume

$$V_p = (1+\Delta\tilde{\varepsilon}/2)\Big|\int\langle\bar{u}(r)\bar{u}(r)\rangle\,\varepsilon_0(r)\mathrm{d}V\Big|[nn^*\langle\bar{u}^*(0)\bar{u}(0)\rangle]^{-1};$$

$$\mu = 3\int\zeta(r)f(r)\,\langle\bar{u}^*(r)\bar{u}(r)\rangle\mathrm{d}V\times$$

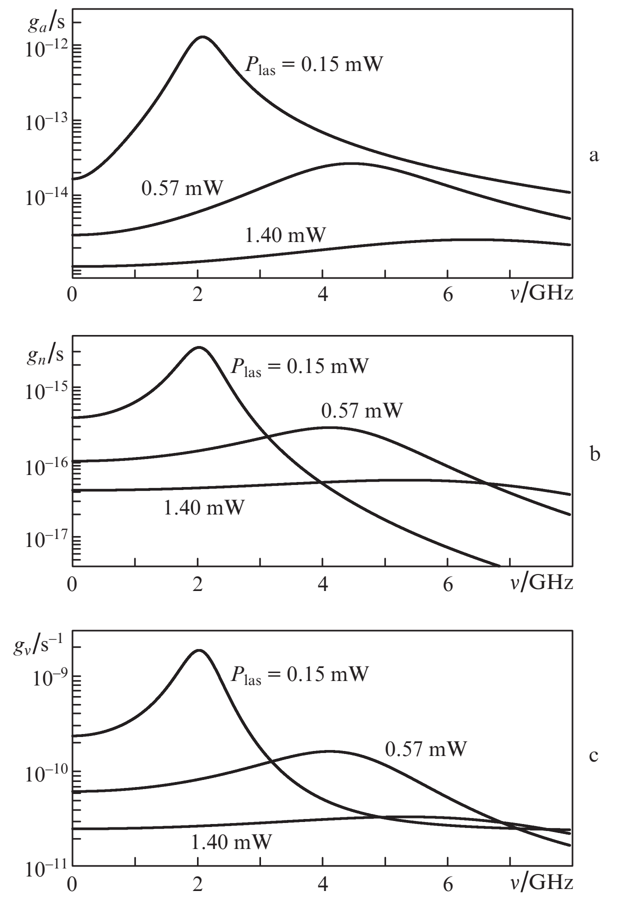


**Figure 1.** Spectra of fluctuations in (a) the laser emission amplitude normalised to the lasing amplitude $A_0$, (b) the electron concentration normalised to the threshold value and (c) the laser frequency.

$$\times\left[\langle\bar{u}^*(0)\bar{u}(0)\rangle\int f(r)\,\mathrm{d}V\right]^{-1} = 0.4;$$

$$K = \frac{\omega_0}{\Delta\omega_{\rm sp}} = 30;$$

energy stored in the cavity

$$\frac{A_0^2|U|(1+\Delta\tilde{\varepsilon}/2)}{8\pi} = \frac{P_{\rm las}}{2\Omega_0\kappa};$$

$\Omega_0 = 250\times10^9$ rad s$^{-1}$; $\theta = 10$; $\tau = 1$ ns; dimensionless (W W$^{-1}$) lasing efficiency $\kappa = 0.34$; $\tilde{R} = 3$; and spontaneous emission power $P = N\hbar\omega/\tau = I_{\rm th}\hbar\omega/e = 0.37$ mA $\times$ 1.45 V = 0.54 mW.

We should also take into account additional information about the nature of the noise source, namely, information about fluctuations resulting from the summation of a huge number of 'small', independent, random impacts. In other words, by virtue of the Moivre–Laplace theorem they are characterised by a normal distribution law. This means that $a(t)$ amplitude and $\tilde{n}(t)$ carrier concentration fluctuations and the phase change (increase in phase in a finite time interval) also have a normal distribution law, with a standard deviation determined by the integral of the corresponding spectral density in the form of relations (26).

The spectrum of frequency fluctuations, $g_\nu(\Omega)$, allows one to find the optical emission spectrum of a single-frequency laser due to spontaneous emission, $S(\omega_0+\Omega)$. According to previous work [21], neglecting $a(t)$ oscillation amplitude fluctuations we obtain

$$S(\omega_0+\Omega) \approx \frac{A_0^2}{2}\int_0^\infty \exp\left[-\frac{\mathcal{F}(t)}{2}\right]\cos(\Omega t)\,\mathrm{d}t, \qquad (27)$$

where the function

$$\mathcal{F}(t) = \varphi^2(t) - \varphi^2(0) \approx 2\int_0^\infty g_\nu(\Omega)\frac{1-\cos(\Omega t)}{\Omega^2}\,\mathrm{d}\Omega$$

characterises time diffusion of the laser oscillation phase (oscillator 'clock' accuracy). Figure 2 shows $\mathcal{F}(t)$ phase diffusion and the $S(\omega_0+\Omega)$ optical spectrum of a vertical cavity diode laser as an example. It is seen that, with increasing laser output power, the fluctuation amplitudes decrease and the resonance frequency rises. This is due to the stabilising effect of laser light as a consequence of the higher stability of the laser operating point and the faster response of the laser system owing to the shorter 'stimulated' transition time. At sufficiently long $t$, the function $\mathcal{F}(t)$ is proportional to $t$, confirming that the motion of the phase has a diffusion character. Numerical calculation using (27) indicates that $S(\omega_0+\Omega)$ can be approximated with good accuracy by a simple expression [12, p. 312]:

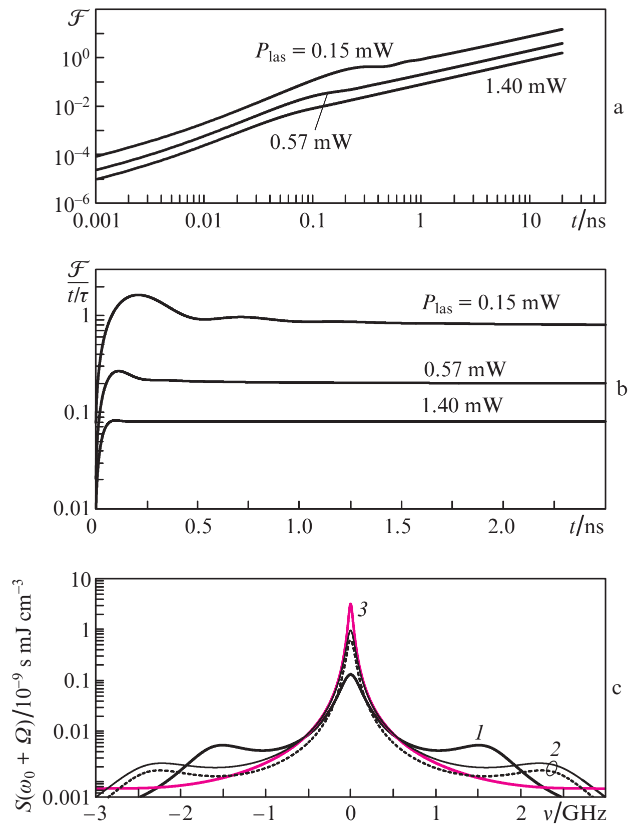


**Figure 2.** (a) Root-mean-square time diffusion of the phase $\mathcal{F}$, (b) effective phase diffusion coefficien $\mathcal{F}(t/\tau)^{-1}$ and (c) laser emission spectrum $S(\omega_0+\Omega)$ at different laser output powers and linewidths (FWHM): (*1*) $P_{\rm las} = 0.0785$ mW, $\Delta\nu_{0.5} = 241.9$ MHz; (*2*) 0.19 mW, 92.7 MHz; (*3*) 0.34 mW, 52.2 MHz; $\nu = \Omega/2\pi$. The dashed line shows the spectrum calculated using formula (28). The calculation parameters are the same as in the case of Fig. 1.

$$S(\omega_0 + \Omega) \approx \frac{A_0^2}{4}\left[\frac{g_v(0)}{(\pi g_v(0)/2)^2 + \Omega^2} + \frac{g_v(\Omega) - g_v(0)}{\Omega^2}\right]. \quad (28)$$

The optical spectrum has the form of a narrow laser line [with a full width at half maximum $\delta\omega \approx \pi g_v(0)$] having a near Lorentzian shape, centre frequency $\omega_0$ and weak, broad wings. The wings have two symmetrically located local maxima (or 'shoulders' at a sufficiently high laser output power) a distance $\sim\Omega_1$ from the laser line, which correspond to the second term in the square brackets in (28). They correspond to the Stokes and anti-Stokes laser light scattering by electron concentration fluctuations.

According to (26c), the laser linewidth $\delta\omega \approx \pi g_v(0)$ is given by

$$\delta\omega \approx \beta N \frac{\hbar\omega_0}{\tau} \frac{\pi(1 + \tilde{R}^2)}{A_0^2 |U| (1 + \Delta\tilde{\varepsilon}/2)} = \beta(1 + \tilde{R}^2)\frac{\mathrm{P}}{\mathrm{Q}}, \quad (29)$$

where

$$\mathrm{P} = N\frac{\hbar\omega_0}{\tau} \text{ and } \mathrm{Q} = A_0^2 |U| \frac{1 + \Delta\tilde{\varepsilon}/2}{\pi}.$$

In (26) the laser frequency $\omega_0$ is thought to be located near the maximum of the spontaneous emission spectrum, so $\rho(\omega_0 + \Omega) \approx \rho(\omega_0 - \Omega) \approx 1$. Formula (29) for $\delta\omega$ has a clear physical meaning if we take into account that P is the total spontaneous emission power and Q is proportional (to within a numerical coefficient) to the laser cavity mode energy and is in qualitative agreement with previously reported results. A more accurate comparison of $\delta\omega$ with other results is difficult to perform because of the significant distinctions between the physical models used and determinations of $\beta$ and the norm $U$ of the mode. The following expression is typically used as a norm:

$$U = \int_V \varepsilon_0(\omega_c, N_0, r) \langle \bar{u}_c(r) \bar{u}_c^*(r) \rangle \mathrm{d}V$$

(by analogy with calculations in quantum-mechanical problems). It includes not only a standing wave but also travelling waves, whereas the expression for $U$ used in this study includes mainly a standing wave in the cavity of a diode laser.

## 6. Conclusions

The present results demonstrate that applying Maxwell's equations to the diode laser leads to a van der Pol-type equation which has a nontrivial solution without any extra emission sources (in addition to stimulated emission), in particular without spontaneous emission sources.

Taking into account spontaneous emission makes the physical model of the diode laser more realistic. Perfectly monochromatic light in the van der Pol model is replaced by quasi-monochromatic light that has a finite spectral width due to spontaneous emission.

We have found an expression for the spontaneous emission factor that quantifies the effect. The same expression is often used for this factor in different problems. It is worth noting here that the expression obtained in this study for the factor of spontaneous emission to a cavity mode differs significantly from that found e.g. by Blokhin et al. [20] for the factor of spontaneous emission to one transverse mode of a travelling wave. This lends support to the idea put forward in discussion in Ref. [20] that a particular expression for this factor depends on the problem in which it is to be employed.

We have obtained for the first time expressions for modified $\tilde{\Gamma}$ and $\tilde{R}$ optical confinement factors and the amplitude–phase coupling constant for a laser cavity.

Expressions have been found for the spectral densities of phase and amplitude fluctuations and the spectral density of electron concentration fluctuations. All these and other quantitative characteristics, including the optical spectrum, have been expressed through diode laser parameters that can be found from independent experiments or calculated if the diode cavity configuration is known.

Lasing and lasing threshold retain their meaning with and without spontaneous emission. Clearly, spontaneous emission leads to fluctuations in all dynamic quantities, which can be viewed as lasing threshold 'broadening'. But this is dynamic 'broadening' (fluctuating in time), rather than some constant difference between the gain and cavity loss, in contrast to what is assumed in phenomenological 'asymptotic threshold' models [7, 8] using rate equations.

Another fundamental distinction between the present results and results obtained in the 'asymptotic threshold' models is related to statistical properties of light. As pointed out above, the lasing amplitude probability density in our model has a normal distribution law around its average $A_0$. The 'asymptotic threshold' model for spontaneous emission amplified to the same power and having the same spectrum always gives a Rayleigh distribution for its amplitude. These are absolutely different distributions with different consequences, e.g. in Brown–Twiss interferometry experiments.

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