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Study of the behaviour of the light field orbital angular momentum upon astigmatic mode conversion

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Abstract. **We consider the behaviour of the orbital angular momentum of the light field as a superposition of Hermite–Gaussian beams upon astigmatic mode conversion. An analytical expression is obtained for the orbital angular momentum of such fields. Expressions are found for various linear combinations of Hermite–Gaussian beams. The astigmatic mode conversion of the initial light fields that are equivalent from the point of view of the orbital angular momentum is shown to lead to significantly different final results.**

Keywords: *coherent optics, spiral light beams, orbital angular momentum.*

1. Introduction

Mode conversion as a means of changing the orbital angular momentum (OAM) of a light field has been known for a long time (see [1] and references therein). A distinctive feature of this conversion is the fact that the OAM in this case dramatically changes as a function of the initial field form.

Abramochkin and Volostnikov [2] theoretically investigated and experimentally implemented astigmatic mode conversion (AMC) of various Hermite–Gaussian beams. An optical system consisting of spherical and cylindrical lenses and performing astigmatic conversion is described in detail in [1]. It was shown in [3, 4] that linear combinations of the Hermite–Gaussian modes can have a significant angular momentum. The asymptotic behavior of such light fields was considered in [4]. Of interest is the question: How does the OAM of a linear combination of Hermite–Gaussian modes change upon AMC?

Of course, as an integral feature, the angular momentum is not a complete characteristic of the light field; however, it is known that in the case of AMC, the change in the orbital momentum depends on the form of the initial field. For example, a field that does not have an OAM can gain it, and vice versa, a field with an angular momentum can lose it as a result of conversion. It is characteristic that the initial Hermite–Gaussian beam with an astigmatic phase weighting function has the same OAM as the resulting Laguerre–Gaussian beam. Nevertheless, it is obvious that the initial field is devoid of optical vortices. It was shown in [3, 4] that linear combinations of Hermite–Gaussian modes can contain optical vortices and have a nonzero OAM.

Examples of nonrotating light fields with a nonzero OAM that do not rotate during propagation can also be found in [4, 5]. Work [6] is devoted to the relationship between the OAM and the rotation of the light field during its propagation. At first glance, it seems somewhat unusual that the sum of fields that do not separately possess an orbital angular momentum does possess this moment. However, it is clear that the system of Hermite–Gaussian functions is complete and orthogonal, and any light field with a finite energy can be represented as their superposition. The meaning of this becomes clearer if we consider the Laguerre–Gaussian mode with an OAM equal to –1.

The aim of this paper is to study the behaviour of the OAM of characteristic light fields before and after astigmatic conversion. Expressions are obtained for the OAM upon AMC of characteristic light fields, which are a linear combination of Hermite–Gaussian beams.

2. OAM of the light field

Consider a plane-polarised field E defined by the expressions

$$E_x = F(x, y, l)\exp(\mathrm{i}kl - \mathrm{i}\omega t),$$

$$E_y = 0,$$

$$E_l = g(x, y, l)\exp(\mathrm{i}kl - \mathrm{i}\omega t).$$

From Maxwell's equation div$\boldsymbol{E} = 0$, we find in the paraxial approximation the relationship for the longitudinal and transverse components of the electric vector:

$$g(x, y, l) \approx \frac{\mathrm{i}}{k}\frac{\partial F}{\partial x}.$$

From Maxwell's equation

$$\boldsymbol{B} = \frac{1}{\mathrm{i}k}\mathrm{rot}\boldsymbol{E},$$

we obtain the components of the magnetic field:

$$B_x \approx 0,$$

$$B_y \approx F(x, y, l)\exp(\mathrm{i}kl - \mathrm{i}\omega t),$$

$$B_l \approx \frac{\mathrm{i}}{k}\frac{\partial F}{\partial x}\exp(\mathrm{i}kl - \mathrm{i}\omega t).$$

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The time-averaged density of the OAM along the l axis is determined by the expression

$$M_l = \frac{1}{8\pi c}\mathrm{Re}[\boldsymbol{r},[\varepsilon\boldsymbol{E},\bar{\boldsymbol{B}}]]_l$$

where ε is the dielectric constant of the medium. Substituting the components of the electric and magnetic fields into it, we obtain

$$\begin{aligned} M_l &= \mathrm{Re}[-x(\varepsilon E_x\bar{B}_l - \varepsilon E_l\bar{B}_x) - y(\varepsilon E_y\bar{B}_l - \varepsilon E_l\bar{B}_y) \\ &= \frac{\varepsilon}{8\pi ck}\mathrm{Im}\left(x\bar{F}\frac{\partial F}{\partial y} - y\bar{F}\frac{\partial F}{\partial x}\right) \\ &= -\frac{\varepsilon}{8\pi ck}\mathrm{Im}\left(xF\frac{\partial \bar{F}}{\partial y} - yF\frac{\partial \bar{F}}{\partial x}\right), \end{aligned} \quad (1)$$

$$L_l = \iint M\mathrm{d}x\mathrm{d}y, \quad E = \iint F\bar{F}\mathrm{d}x\mathrm{d}y.$$

Then we substitute the representation of the field in the form of a superposition of the Laguerre–Gauss modes in these expressions, using their orthogonality:

$$\begin{aligned} F(x,y) &= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} c_{nm}LG_{nm}(x,y), \\ \frac{L_l}{E} &= \sum_{n,m=0}^{\infty} m\frac{(n+m)!(|c_{nm}|^2 - |c_{n-m}|^2)}{2^m n!} \\ &\times\left[\sum_{n,m=0}^{\infty} m\frac{(n+m)!(|c_{nm}|^2 + |c_{n-m}|^2)}{2^m n!}\right]^{-1}. \end{aligned} \quad (2)$$

3. AMC of light fields as a superposition of Hermite–Gaussian beams and the resulting OAM

Astigmatic mode conversion A in mathematical form can be represented by the formula:

$$\begin{aligned} A(\alpha, HG_{nm}(x,y)) &= G_{nm}(\alpha) \\ &= \exp[\mathrm{i}\varphi(x,y,\alpha)/8]\exp[\mathrm{i}\pi(n+m)/4] \\ &\times\iint_{R^2}\exp[-\mathrm{i}(x\xi + y\eta) + \mathrm{i}\varphi(\xi,\eta,\alpha)]HG_{nm}(\xi,\eta)F(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta, \end{aligned}$$

where

$$HG_{nm} = H_{nm}(x,y)\exp\left(-\frac{x^2+y^2}{2}\right)$$

are the Hermite–Gaussian beams;

$$\varphi(\xi,\eta,\alpha) = (\xi^2 - \eta^2)\cos(2\alpha) + 2\xi\eta\sin(2\alpha);$$

and the parameter α has the physical meaning of the angle of rotation of an optical system consisting of spherical and cylindrical lenses and implementing astigmatic conversion relative to the optical axis.

For the Hermite–Gaussian modes, this conversion transforms them into Laguerre–Gaussian modes (at $\alpha = \pi/4$) [3] or into Hermite–Gaussian modes (at $\alpha = 0$). The conversion into the Hermite–Gaussian mode is nontrivial, i.e. the resulting Hermite–Gaussian mode has an additional phase factor [3]:

$$\begin{aligned} A(\pi/4, HG_{nm}(x,y)) &= G_{nm}(\pi/4) \\ &= \exp(\mathrm{i}xy/4)\exp[\mathrm{i}\pi(n+m)/4] \\ &\times\iint_{R^2}\exp[-\mathrm{i}(x\xi + y\eta)]HG_{nm}(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta \\ &= \begin{cases} (-1)^m 2^n m!LG_{mn-m}\left(\frac{x}{2\sqrt{2}},\frac{y}{2\sqrt{2}}\right) & \text{at } n \geqslant m, \\ (-1)^n 2^m n!LG_{nm-n}\left(\frac{y}{2\sqrt{2}},\frac{x}{2\sqrt{2}}\right) & \text{at } n \leqslant m, \end{cases} \\ A(0, HG_{nm}(x,y)) &= G_{nm}(0) = \exp[\mathrm{i}(x^2 - y^2)/4] \\ &\times\exp[\mathrm{i}\pi(n+m)/4]\iint_{R^2}\exp[-\mathrm{i}(x\xi + y\eta) + \mathrm{i}(\xi^2 - \eta^2)] \\ &\times HG_{nm}(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta = (-\mathrm{i})^m HG_{nm}(x,y). \end{aligned} \quad (3)$$

Let the initial fields have the form

$$\begin{aligned} F_1 &= HG_n(x)HG_{n+1}(y), \\ F_2 &= HG_n(x)HG_{n+1}(y) + HG_{n+1}(x)HG_n(y). \end{aligned} \quad (4)$$

Let us denote their modal astigmatic conversions (at $\alpha = 0$) as follows:

$$\begin{aligned} A(\alpha, F_1) &= G_1(\alpha) = A(\alpha, HG_n(x)HG_{n+1}(y)), \\ A(\alpha, F_2) &= G_2(\alpha) = A(\alpha, HG_n(x)HG_{n+1}(y) \\ &+ HG_{n+1}(x)\,HG_n(y)) = G_{2\,n\,n+1}(\alpha) + G_{2\,n+1\,n}(\alpha). \end{aligned} \quad (5)$$

These fields were chosen because they have similar properties: both are real and structurally stable during propagation, but their astigmatic conversions have completely different OAMs.

It is interesting and important to note that the modal astigmatic conversion $G_2(\alpha)$ at $\alpha = 0$ coincides, up to a constant factor $(-\mathrm{i})^{n+1}$, with the superposition of fields given in [4, 5]; therefore, the specific OAM will be the same:

$$\left.\frac{L_l(G_{n\,n+1}(\alpha) + G_{n+1\,n}(\alpha))}{E(G_{n\,n+1}(\alpha) + G_{n+1\,n}(\alpha))}\right|_{\alpha=0} = -(n+1).$$

This property is also valid for other cases considered in [4, 6]. For example, let the light field have the form

$$\begin{aligned} F(x,y) &= HG_n(x)HG_{n+1}(y) + A_1HG_{n+1}(x)HG_n(y) \\ &+ A_2HG_{n+2}(x)HG_{n-1}(y). \end{aligned}$$

Its AMC is expressed as

$$\begin{aligned} G(\alpha = 0, x, y) &= (-\mathrm{i})^{n+1}\,[HG_n(x)HG_{n+1}(y) \\ &+ \mathrm{i}A_1HG_{n+1}(x)HG_n(y) + \mathrm{i}^2A_2HG_{n+2}(x)HG_{n-1}(y)]. \end{aligned}$$

Thus, the problem of mode astigmatic conversion of light fields of form (5) has been reduced to a problem that was studied in detail in [4, 6]. It can be seen from the above expressions that a real light field can acquire a substantially nonzero OAM in the case of AMC.

Consider the astigmatic conversion of the above light fields. Let the field have the form

$$F = HG_n(x)HG_{n+1}(y) + \mathrm{i}HG_{n+1}(x)HG_n(y).$$

AMC in this case is of interest at $\alpha = \pi/4$, since at $\alpha = 0$ mode conversion yields $L = 0$. After AMC we have

$$A(\pi/4, F) = (-1)^n[2^{n+1}n!LG_n^{(-1)}(x, y) + \mathrm{i}2^{n+1}n!LG_n^{(1)}(x, y)].$$

The specific orbital momentum of such a field is also zero:

$$\frac{L_l}{E} = 0.$$

Consider now the field [6]

$$F = HG_n(x)HG_n(y) + \mathrm{i}HG_{n+1}(x)HG_{n-1}(y).$$

From work [4] it follows that this combination has the same asymptotic value of the OAM as field (2), but for AMC we obtain in this case the expression

$$A(\pi/4, F) = (-1)^n 2^n n!LG_n^{(0)}(x, y) + (-1)^{n-1}\mathrm{i}2^{n+1}(n-1)!LG_{n-1}^{(2)}(x, y).$$

Accordingly, the specific OAM is expressed as

$$\frac{L_l}{E} = -\frac{2(n-1)!(n+1)!}{(n!)^2 + (n-1)!(n+1)!} = -\left[\frac{2}{n/(n+1)} + 1\right].$$

The asymptotic value of the OAM at $n \to \infty$ is –1. Thus, with AMC, the equality of OAMs for these two cases is not satisfied.

Figure 1 shows intensity and phase distributions of field (5) for $n = 4$ and the result of its AMC for $\alpha = \pi/4$.

Now let the light field be a superposition of three Hermite–Gaussian modes (the weight coefficients are borrowed from [6]):

$$F(x, y) = HG_n(x)HG_{n+1}(y) + \mathrm{i}A_1HG_{n+1}(x)HG_n(y) + \mathrm{i}^2A_2HG_{n+2}(x)HG_{n-1}(y), \quad (6)$$

$$A_1 = \sqrt{2},\ A_2 = 1.$$

Let us find its AMC:

$$A(\alpha, F) = (-1)^n 2^{n+1}n!LG_n^{(-1)}(x, y) + (-1)^n 2^{n+1}A_1 n!LG_n^{(1)}(x, y) + \mathrm{i}^2(-1)^{n-1}2^{n+2}A_2(n-1)!LG_{n-1}^{(3)}(x, y).$$

The specific OAM of such a field for the case considered in [6] is determined by the expression

$$\frac{L_l}{E} = \frac{\frac{1}{2}n!(n+1)! + \frac{3}{2}(n-1)!(n+1)!}{\frac{1}{2}n!(n+1)! + n!(n+1)! + \frac{1}{2}(n-1)!(n+1)!} = \left(\frac{1}{2} + \frac{3}{2}\frac{n+2}{n}\right)\left(\frac{3}{2} + \frac{1}{2}\frac{n+2}{n}\right)^{-1}.$$

The asymptotic value of the OAM at $n \to \infty$ is 1. Figure 2 shows intensity and phase distributions of field (6) at $n = 4$ and the result of its AMC ($\alpha = \pi/4$).

Let us now consider a ‘symmetric’ combination (the weights are borrwed from [6]), which is the result of mode conversion of a symmetric combination of fields:

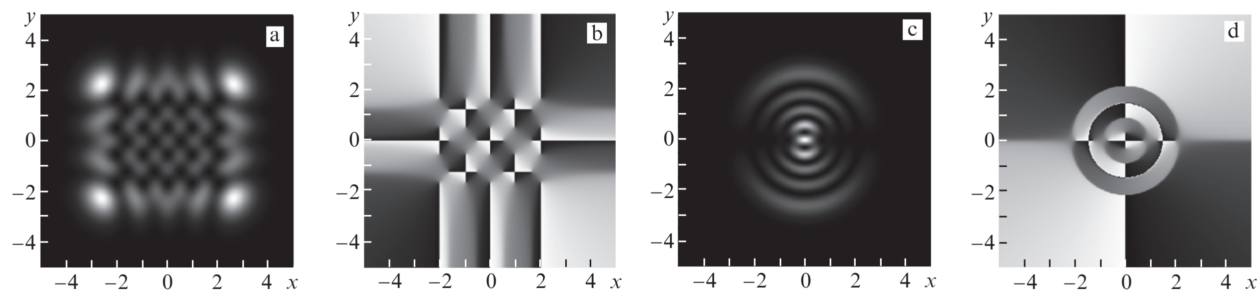

Figure 1. Distributions of (a, c) amplitude and (b, d) phase of field (5) before (a, b) and after (c, d) AMC ($\alpha = \pi/4$).

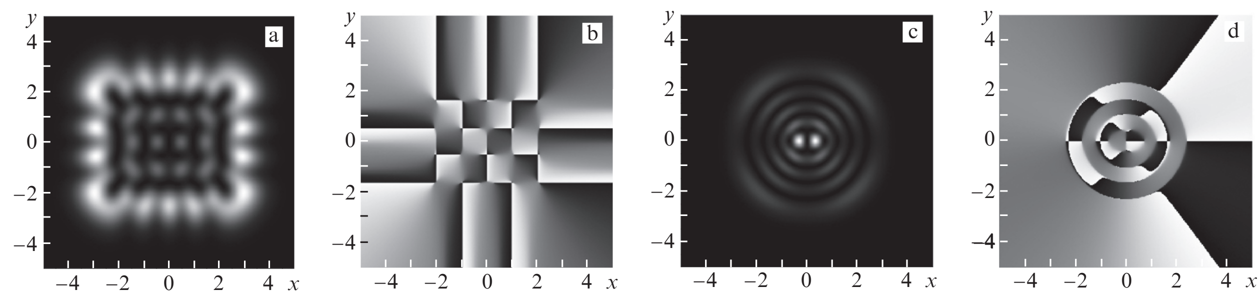

Figure 2. Distributions of (a, c) amplitude and (b, d) phase of field (6) before (a, b) and after (c, d) AMC ($\alpha = \pi/4$).

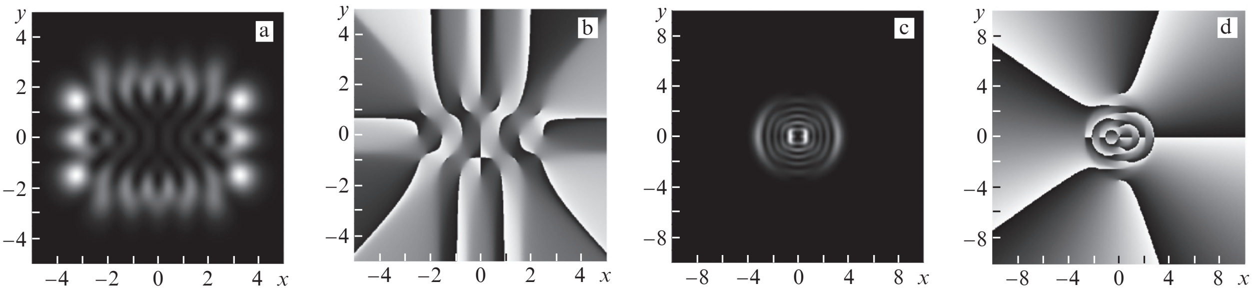

Figure 3. Distributions of (a, c) amplitude and (b, d) phase of field (8) before (a, b) and after (c, d) AMC ($\alpha = \pi/4$).

$$F(x,y) = -\mathrm{i}A_{-1}HG_{n-1}(x)HG_{n+2}(y) + HG_n(x)HG_{n+1}(y) + \mathrm{i}A_1HG_{n+1}(x)HG_n(y). \quad (7)$$

Its AMC ($\alpha = \pi/4$) is determined by the expression

$$A(\alpha,F) = -\mathrm{i}(-1)^{n-1}A_{-1}2^{n+2}(n-1)!\,LG_{n-1}^{(-3)}(x,y) + 2^{n+1}(-1)^n n!LG_n^{(-1)}(x,y) + \mathrm{i}2^{n+1}(-1)^n n!A_1LG_n^{(1)}(x,y).$$

In this case, the OAM will be as follows:

$$\frac{L_l}{E} = \frac{-5/4 + 1/4}{1/2 + 1/4 + 1/4} = -1.$$

Finally, we consider a superposition of four Hermite–Gauss modes (the weights are borrowed from [6]):

$$F = HG_n(x)HG_{n+1}(y) + \mathrm{i}A_1HG_{n+1}(x)HG_n(y) + \mathrm{i}^2A_2HG_{n+2}(x)HG_{n-1}(y) + \mathrm{i}^3A_3HG_{n+3}(x)HG_{n-2}(y),$$

$$n \geqslant 2,\ A_1 = A_2,\ A_3 = 1,\ A_2 = (1+\sqrt{5})/2. \quad (8)$$

Let us find the result of its AMC:

$$A(\alpha,F) = (-1)^n2^{n+1}n!LG_n^{(-1)}(x,y) + \mathrm{i}A_12^{n+1}(-1)^n n!G_n^{(1)}(x,y) + \mathrm{i}^2(-1)^{n-1}A_22^{n+2}(n-1)!LG_n^{(3)}(x,y) + \mathrm{i}^3(-1)^{n-2}A_32^{n+3}(n-2)!LG_n^{(5)}(x,y). \quad (9)$$

For the OAM we obtain the expressio

$$\frac{L_l}{E} = \frac{2 + 2\left[\frac{1}{2}(1+\sqrt{5})\right]^2}{1 + \left[\frac{1}{2}(1+\sqrt{5})\right]^2} = 2.$$

Figure 3 shows intensity and phase distributions of field (8) at $n = 4$ and the result of its AMC. Let us compare this result with the result corresponding to the symmetric combination:

$$F(x,y) = -\mathrm{i}A_{-2}HG_{n-1}(x)HG_{n+2}(y) + A_{-1}HG_n(x)HG_{n+1}(y) + \mathrm{i}A_1HG_{n+1}(x)HG_n(y) + \mathrm{i}^2A_2HG_{n+2}(x)HG_{n-1}(y).$$

The AMC for this field is described by the expression

$$A(\alpha,F) = -\mathrm{i}(-1)^{n+2}2^{n-1}(n-1)!A_{-2}LG_{n-1}^{(-3)}(x,y) + A_{-1}2^{n+1}(-1)^{n+1}n!LG_n^{(-1)}(x,y) + \mathrm{i}A_12^{n+1}(-1)^n n!LG_n^{(1)}(x,y) + \mathrm{i}^2A_22^{n+2}(-1)^{n-1}(n-1)!\,LG_{n-1}^{(3)}(x,y),$$

and the specific OAM will be zero:

$$\frac{L_l}{E} = -6A_{-2}^2 + 6A_2^2 = 0.$$

It was shown in [4] that for field (8) and for the symmetric combination, the asymptotic OAMs are the same. From the above consideration, it can be seen that for these combinations the equality of OAMs after AMC also does not hold.

4. Conclusions

Thus, we have studied the change in the OAM in the case of the AMC. It has been found that the OAM changes radically. Moreover, even fields with the same OAM acquire completely different values in the case of the AMC. For example, this is true for real structurally stable light fields.

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