Quantum Electronics **50** (2) 187–194 (2020) ©2020 Kvantovaya Elektronika and Turpion Ltd

EXTREMELY SHORT LASER PULSES

https://doi.org/10.1070/QEL17153

Energy density and spectrum of single-cycle and sub-cycle electromagnetic pulses

I.A. Artyukov, A.V. Vinogradov, N.V. D'yachkov, R.M. Feshchenko

Abstract. **Based on the exact solution of the Maxwell equations in the form of a collapsing spherical vector wave specified by an arbitrary function of time, we have calculated the maximum energy density that can be achieved when focusing extremely short pulses of various shapes. It is shown that our earlier formula expressing the maximum energy density in terms of the spectrum parameters for Gaussian quasi-monochromatic pulses is approximately (with an accuracy of ±40 %) valid for the main types of extremely short pulses.**

Keywords: *single-cycle and sub-cycle electromagnetic pulses, collapsing spherical vector wave, Maxwell equations.*

1. Introduction

The generation and application of ultrashort laser pulses with a small number of field cycles have attracted attention for two decades [1–3]. These pulses are widely used in nonlinear optics, in the study of fast processes, in laser technologies, and in a number of other fields. In addition, the spatio-temporal couplings are most pronounced for them, namely, the dependence of the temporal and spectral characteristics of coherent radiation on the position (coordinate) of the detector [4–7]. In our previous paper [8], the field evolution in collapsing quasi-monochromatic electromagnetic pulses was considered. In this work, following [8], by an example of the simplest exact solution of the Maxwell equations for a finite-energy beam, we consider the evolution of collapsing ultrashort pulses whose duration is on the order of the light wave period.

The second section is devoted to a review of works on finding exact solutions of the Maxwell equations in vacuum. Section 3 gives a brief derivation of one of the solutions that describes an electromagnetic pulse with the shape of a collapsing shell and discusses its main properties, including the ratio of the maximum energy density to the total pulse energy. Further, in Section 4, for nonmonochromatic beams, the average frequency and line width (dispersion) are determined, which are then used in this section and in Section 6 to describe quasi-monochromatic pulses, as well as single-cycle and half-cycle ones and their focusing parameters. In Section 5, an estimate of the energy density at the centre of a collapsing pulse for comparison is obtained from thermodynamic considerations and using the Fresnel integral. The Appendix considers the property of a three-dimensional wave equation, with the help of which the method of the present work can be extended to collapsing electromagnetic shells of a more complex geometry.

I.A. Artyukov, A.V. Vinogradov, N.V. D'yachkov, R.M. Feshchenko Lebedev Physical Institute, Russian Academy of Sciences, Leninsky prosp. 53, 119991 Moscow, Russia; e-mail: vinograd@lebedev.ru

Received 14 October 2019
Kvantovaya Elektronika **50** (2) 187–194 (2020)
Translated by V.L. Derbov

2. On exact solutions of the electromagnetic field equations

Suppose that there is a laser pulse with energy $\mathcal{E}$. What is the maximum density of electromagnetic energy $\epsilon_{\rm m}$ that can be obtained by focusing it? How does focusing change its shape and spectrum? Such questions require the development of theoretical models based on solutions of the Maxwell equations, which satisfy two conditions: the localisation of the solution, which means finiteness of the total energy $\mathcal{E}$ equal to the integral of the energy density $\epsilon(\boldsymbol{r}, t)$ over the entire space, and the impossibility of representing the field as a product of spatial and temporal parts. We note immediately that an ordinary Gaussian beam, as well as any monochromatic beam, do not satisfy these conditions. However, their superposition can be used to describe beams having a finite spectral width and energy.

The development of laser physics and, especially, lasers generating ultrashort pulses containing a small number of field periods, gave rise to rapid growth of interest in studying the exact solutions of the Maxwell equations for nonmonochromatic beams, which, unlike Gaussian beams, can be directly used to solve the above problems of ultrashort laser pulse physics.

Let us list the main approaches to obtaining exact solutions of the electromagnetic field equations, which are widely used in optics and laser physics. The best known are exact solutions in the form of Gaussian beams and Fresnel integrals of a parabolic wave equation, which itself is approximate. Therefore, such approaches are applicable, strictly speaking, only for monochromatic and narrowly directed (paraxial) beams*. To model nonparaxial monochromatic beams, complexified (obtained by shifting one of the coordinates to the complex plane) spherical waves and other exact solutions of the Helmholtz equation describing harmonic (monochromatic) electromagnetic fields are used [9–11]. However, for the above reasons, the parabolic wave equation and the Helmholtz equation are not suitable for describing the effects considered in this paper, and hereinafter we will discuss only

* Despite this fact, the parabolic equation and the beams associated with it are widely used in practical optics and physics of linear and nonlinear wave processes. For more details about the applicability conditions for the parabolic wave equation, see [9]. The subject of this work is beyond the scope of discussing these conditions.

the complete system of Maxwell's vector equations. In this case, most of the work on finding exact solutions relies on the deep connection of the Maxwell equations with the scalar wave equation, known since the beginning of the last century [12]. In particular, any solution to the latter makes it possible to construct an exact solution of Maxwell's equations using elementary vector operations (see below)*. Due to this fact, the problem is reduced to finding the exact solutions of the scalar wave equation. The simplest spherically symmetric solution [14, 15] is used in our work (see Section 3). The exact solution of the equations for an electromagnetic field with finite energy based on it is no longer spherically symmetric, but possesses a more complex symmetry, corresponding to the angular dependence of the characteristics of the radiation of a dipole.

Probably, for the first time the exact solutions of the Maxwell equations were used to describe ultrashort laser pulses by Helwarth et al. [16–18]. The authors of these works used the method based on the scalar wave equation that became standard by that time, but instead of the spherically symmetric solution, they used a more complex one having cylindrical symmetry, found earlier in [11]. It served as a 'generating function' for beams corresponding to exact non-singular solutions of Maxwell's equations and having a doughnut shape in focus. The results of the calculation were used to describe single-cycle laser pulses and their spatiotemporal correlations, the acceleration of relativistic electrons by them, and experiments with terahertz radiation. The possibilities of obtaining such pulses in waveguides and resonators with curved mirrors were discussed.

In Ref. [19], which is a development of [20], the structure of focused laser beams is studied in order to observe quantum electrodynamic effects. It also applies the scalar wave equation method with a generating function having axial symmetry. However, instead of the standard operator [12, 13], a new operator is proposed that transforms this function into the solution of Maxwell's equations. The structure of focused harmonic and sub-cycle pulses is studied. The research results are used in calculating the threshold intensity at which electron–positron pairs are created in the laser field [21].

The simplest, in the sense of being described by elementary functions, light pulse with finite energy can be imagined as follows. A spherical shell of arbitrary shape, filled with a field, contracts to the centre, passes through it, interfering with itself for some time, and then continues to expand, going to infinity. In this case, the shape of the shell changes slightly, except for the moment of collapse, after which it is restored. The corresponding solution is discussed in [8, 22] and in the present paper. To derive a nonsingular solution, Gonoskov et al. [22] introduce the Hertz vector of a point source, and then, as in Ref. [19], add the retarded and advanced potentials following [23, 24]. The calculation results are used to analyse the structure of the field, estimate its maximum value and the volume occupied by the field during focusing. A method is proposed for obtaining such beams in '4π-focusing systems' with parabolic mirrors. In Ref. [8], the same solution is found by the standard method of the scalar wave equation, but the temporal structure of the pulse is different. A significant drawback limiting the practical application of the results [8, 22] is the lack of a parameter responsible for the divergence. Nevertheless, they allow us to relate the maximum field at the moment of compression to the total energy and spectrum of the incident pulse [8], as well as to trace the evolution of the pulse shape during focusing. This circumstance indicates the fundamental possibility of the formation of the temporal structure of the pulse to achieve maximum fields in focus [22].

The theory of electromagnetic pulses, their invariants and examples is the subject of monograph [25]. A large number of exact solutions of the scalar wave equation are presented in review [9] and in the later paper [26]. In accordance with the foregoing, they can be used to build free electromagnetic fields. One more approach that is regular can be developed using the classification of solutions of the wave equation and Maxwell equations based on representations of the group of rotations and spherical vectors [27, 28]. The result presented in the Appendix also evidences in favour of this.

3. Collapsing shell – electromagnetic pulse with finite energy

It is known that the Maxwell equations for an electromagnetic field in free space can be written as a system of equations for the vector potential $\boldsymbol{A}$:

$$\Delta \boldsymbol{A} = \frac{1}{c^2}\ddot{\boldsymbol{A}}, \tag{1a}$$

$$\mathrm{div}\boldsymbol{A} = 0, \tag{1b}$$

where c is the velocity of light in vacuum. In this case, the vectors of the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{H}$) fields are expressed in terms of the potential $\boldsymbol{A}$ as follows:

$$\boldsymbol{E} = -\frac{1}{c}\dot{\boldsymbol{A}}, \quad \boldsymbol{H} = \mathrm{rot}\boldsymbol{A}. \tag{2}$$

The solutions of system (1) interesting for us and the fields (2) corresponding to them have a general form:

$$\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{l}\times\nabla u(\boldsymbol{r},t), \quad \boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{l}\times\nabla u(\boldsymbol{r},t),$$
$$\boldsymbol{H} = \boldsymbol{l}\Delta u(\boldsymbol{r},t) - (\boldsymbol{l}\nabla)\nabla u(\boldsymbol{r},t), \tag{3}$$

where $\boldsymbol{l}$ is an arbitrary unit axial vector, and $u(\boldsymbol{r},t)$ is an arbitrary solution of the scalar wave equation

$$\Delta u = \frac{1}{c^2}\ddot{u}. \tag{4}$$

If we take the solution of Eqn (4) in the form of a plane wave, then it is easy to verify that substituting it into Eqns (3) and (2) we will get a solution of Maxwell's equations (1) in the form of a plane wave with linear polarisation, whose vector is perpendicular both to the vector $\boldsymbol{l}$, and the direction of wave propagation. However, the solution in the form of a plane wave is not suitable for the mentioned problem, since it has infinite energy.

To solve this problem, it is possible to use a spherically symmetric solution of Eqn (4) without a singularity at zero:

$$u(\boldsymbol{r},t) = \frac{f(ct+|\boldsymbol{r}|) - f(ct-|\boldsymbol{r}|)}{|\boldsymbol{r}|}, \tag{5}$$

*The other (in a certain sense, converse) statement is also true, namely, the Whittaker theorem [12, 13]: Any free electromagnetic field can be expressed in terms of two solutions of the scalar wave equation. The theorem is more elegantly formulated as follows: Any free electromagnetic field is a superposition of two fields, each of which is constructed on the basis of the exact solution of the scalar wave equation.

where $f(x)$ is a real, smooth function, rapidly decreasing for large x. Substitution of this expression into Eqn (3) gives the following expressions for the electric and magnetic fields:

$$\boldsymbol{H} = \boldsymbol{l}\left(\Delta u - \frac{1}{r}\frac{\partial u}{\partial r}\right) - \boldsymbol{n}(\boldsymbol{ln})r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right),$$

$$\boldsymbol{E} = \frac{\boldsymbol{n}\times\boldsymbol{l}}{c}\frac{\partial^2 u}{\partial t\partial r}, \quad r = |\boldsymbol{r}|, \quad \boldsymbol{n} = \frac{\boldsymbol{r}}{r}. \tag{6}$$

It is logical to call the point $\boldsymbol{r} = 0$ the point of the wave collapse. Expressions (6) are greatly simplified in two limiting cases: for points infinitely remote in time and space and for the collapse point itself. In the first case (for $ct \to \infty$, $r = ct - x \to \infty$, x being a variable whose modulus does not exceed an order of magnitude of the pulse length), we have

$$\boldsymbol{E} = \frac{\boldsymbol{n}\times\boldsymbol{l}}{r}g(ct-r), \quad \boldsymbol{H} = \frac{\boldsymbol{l}-\boldsymbol{n}(\boldsymbol{ln})}{r}g(ct-r), \tag{7}$$

and in the second case (for $r = 0$) we obtain

$$\boldsymbol{E} = 0, \quad \boldsymbol{H} = \frac{4}{3}\boldsymbol{l}g'(ct), \tag{8}$$

where

$$g(x) = f''(x) \tag{9}$$

is a new arbitrary function in terms of which the total pulse energy $\mathcal{E}$ and the energy density at the centre $\epsilon(\boldsymbol{r} = 0, t)$ can be expressed. The corresponding formulae are given in Ref. [8].

Expressions (7) and (8) demonstrate the effect of spatio-temporal coupling: In the process of propagation to the centre, the shape of the collapsing impulse transforms into the shape of its derivative. They also make it possible to obtain a number of useful relations, for example, the relation between the total pulse energy $\mathcal{E}$ and the maximum value of energy density $\epsilon_{\rm m}$ at the collapse point:

$$\frac{\epsilon_{\rm m}}{\mathcal{E}} = \frac{\max[g'(x)]^2}{3\pi\int_{-\infty}^{\infty} g^2(x)\,{\rm d}x}. \tag{10}$$

Here, to calculate the total pulse energy, the Poynting theorem was used. Expressions (7) and (8) make it possible to obtain a simple relation for the pulse spectrum shapes at the collapse point $[F_0(\omega)]$ and far from it

$$F_0(\omega) \propto \left|\int_{-\infty}^{\infty} g'(x)\exp\left(-\frac{{\rm i}\omega x}{c}\right){\rm d}x\right|^2$$
$$= \left(\frac{\omega}{c}\right)^2\left|\int_{-\infty}^{\infty} g(x)\exp\left(-\frac{{\rm i}\omega x}{c}\right){\rm d}x\right|^2 \propto \omega^2 F_\infty(\omega). \tag{11}$$

Formula (10) is the most general, but it does not allow expressing the above ratio in terms of generally accepted pulse parameters: duration, frequency, and width of the spectrum. Indeed, such expressions should depend on the specific shape (type) of the incident pulse, some of which will be considered below. To begin with, we will specify the choice of the main universal temporal and spectral characteristics of the pulse.

4. Quasi-monochromatic Gaussian pulse

When considering real-valued pulses of arbitrary shape $g(t)$, we choose the root of the time variance σ_t as a temporal characteristic and call it the duration:

$$\sigma_t = \sqrt{\langle t^2\rangle - \langle t\rangle^2}, \quad \langle t\rangle = \frac{\int_{-\infty}^{\infty} tg^2(t)\,{\rm d}t}{\int_{-\infty}^{\infty} g^2(t)\,{\rm d}t},$$

$$\langle t^2\rangle = \frac{\int_{-\infty}^{\infty} t^2 g^2(t)\,{\rm d}t}{\int_{-\infty}^{\infty} g^2(t)\,{\rm d}t}. \tag{12}$$

As the spectral characteristics, we use the centre (average) frequency ω_0

$$\omega_0 = \langle\omega\rangle = \frac{\int_0^{\infty} \omega\,|G(\omega)|^2{\rm d}\omega}{\int_0^{\infty} |G(\omega)|^2{\rm d}\omega}, \tag{13}$$

and spectrum width σ_ω (square root of frequency variance):

$$\sigma_\omega = \sqrt{\langle\omega^2\rangle - \omega_0^2}, \quad \langle\omega^2\rangle = \frac{\int_0^{\infty} \omega^2\,|G(\omega)|^2{\rm d}\omega}{\int_0^{\infty} |G(\omega)|^2{\rm d}\omega}, \tag{14}$$

where

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\exp(-{\rm i}\omega t)\,{\rm d}t \tag{15}$$

is the Fourier transform of the function $g(t)$. We start with the pulse considered in [8, 22], for which the function $g(x = ct)$ is given by the expression

$$g(x) = C\exp(-x^2/a^2)\sin(qx). \tag{16}$$

If

$$qa \gg 1, \tag{17}$$

then the pulse can be called quasi-monochromatic Gaussian, and the duration σ_t, the spectrum width σ_ω, and the centre frequency ω_0 according to (12)–(14) are determined by the relations

$$\sigma_t = a/(4c), \quad \omega_0 = qc, \quad \sigma_\omega = c/a. \tag{18}$$

Substituting function (16) into expression (10) and taking Eqns (18) into account, we obtain [8]

$$\frac{\epsilon_{\rm m}}{\mathcal{E}} = \frac{q^2}{3a}\left(\frac{2}{\pi}\right)^{3/2} = \frac{8}{3}\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3\frac{\sigma_\omega}{\omega_0}. \tag{19}$$

Here $\lambda_0 = 2\pi c/\omega_0$ is the centre wavelength for the considered pulse. Expression (19) has a clear meaning: The ratio of the maximum energy density of a pulse to its total energy is proportional to the relative width of the spectrum and inversely proportional to the volume of the region of space with linear dimensions of the order of λ_0.

5. Approximate estimation of the maximum energy density for the quasi-monochromatic case ('thermodynamic limit')

For a quasi-monochromatic beam, a formula similar to Eqn (19) can be obtained by estimating the maximum attainable brightness of the radiation at the focal point. Let the pulse be a plane wave incident on an ideal focusing system. The entry to this system is circular and has a radius R, and the exit is visible from the focal point at a solid angle Ω (Fig. 1). To assess the brightness of the radiation at the focal point, we use the theorem that it cannot exceed the brightness of the source. As is known, brightness is a concept of geometric optics, while the indicated limit of the energy density is completely determined by the wave properties of radiation. However, this discrepancy can be overcome by replacing the mentioned focusing system with an equivalent one (Fig. 2), in which the paraxial approximation can be used to estimate the 'wave' constraints.

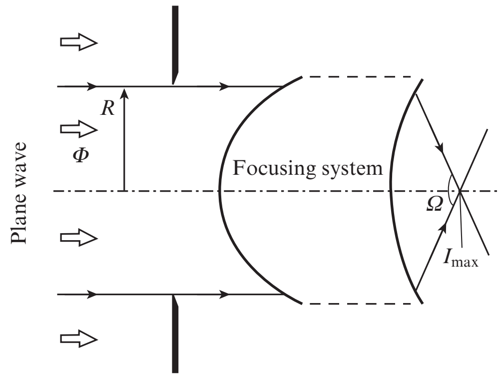


Figure 1. Scheme of plane wave focusing.

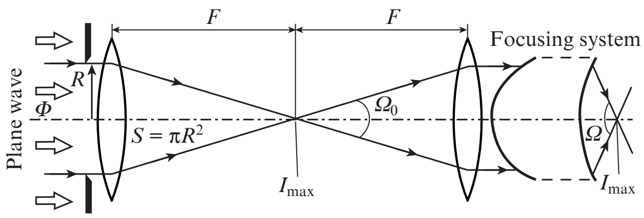


Figure 2. Equivalent supplemented scheme of plane wave focusing.

The supplemented scheme shown in Fig. 2 differs from the initial one (Fig. 1) by the presence of a diaphragm of radius R and two identical lenses with a common focus in front of the focusing system. It is assumed here that

$$1 \gg R/F \gg \lambda_0/R. \tag{20}$$

In the considered case, the beam at the exit of the second lens will practically not differ from the beam at the entrance of the first lens, since the laws of geometric optics are valid when the second inequality (20) is satisfied. Moreover, we can say that our focusing system, together with the second lens, actually creates an image of a diffraction spot located at the point of common focus of the lenses. The maximum radiation intensity in this diffraction spot is easily found using the Fresnel integral, knowing the corresponding maximum radiation flux Φ_{max} through the input diaphragm:

$$I_{0\max} = \frac{\Phi_{\max}}{\pi R^2}\left(\frac{\pi R^2}{\lambda_0 F}\right)^2 = \Phi_{\max}\frac{\Omega_0}{\lambda_0^2}. \tag{21}$$

Here $\Omega_0 = \pi R^2/F^2$ is actually the solid angle in which the radiation from the diffraction spot is concentrated. Thus, the diffraction spot at the focus of the lenses can be considered as an extended radiation source with maximum brightness

$$B_{\max} = \frac{I_{0\max}}{\Omega_0} = \frac{\Phi_{\max}}{\lambda_0^2}. \tag{22}$$

Then, according to the above theorem, the brightness of the radiation at the focal point is limited by value (22). This fact means that for the desired maximum energy density ϵ_m, we can write the following estimate:

$$\epsilon_m = \frac{I_{\max}}{c} \leqslant \frac{\Omega B_{\max}}{c} = 4\pi\frac{\Phi_{\max}}{\lambda_0^2 c}. \tag{23}$$

Here we assume that the maximum solid angle in which the focusing is performed is 4π.

It remains to express the maximum energy flux Φ_{max} through the pulse parameters. For a pulse having the form of (16), with Eqn (17) taken into account, the time dependence of the energy flux $\Phi(t)$ can be represented as

$$\Phi(t) = \Phi_{\max}\exp\left(-\frac{2c^2t^2}{a^2}\right)\sin^2(qct). \tag{24}$$

Then the total energy is

$$\mathcal{E} = \int_{-\infty}^{\infty}\Phi(t)\,dt = \sqrt{\frac{\pi}{8}}\frac{a}{c}\Phi_{\max}. \tag{25}$$

Finally, taking (25) and (18) into account, expression (23) can be presented as

$$\frac{\epsilon_m}{\mathcal{E}} \leqslant \frac{8\sqrt{2\pi}}{\lambda_0^2 a} = \left(\frac{2}{\pi}\right)^{3/2}\frac{q^2}{a} = 8\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3\frac{\sigma_\omega}{\omega_0}. \tag{26}$$

Note that the resulting formula differs from the exact formula (19) by a numerical factor of 3.

6. Ultrashort pulses of various shapes

We now consider the case of ultrashort pulses and try to obtain expressions similar to (19) for them. We note that for these pulses, the average frequency and the average wavelength are not related by such a simple formula as (19) for a quasi-monochromatic pulse. At the same time, the average wavelength calculated by averaging over frequencies can be infinite. In this regard, the characteristic wavelength for the pulse λ_0 will be defined using the average frequency ω_0 in the same way as in the quasi-monochromatic case:

$$\lambda_0 = \frac{2\pi c}{\omega_0}, \quad \omega_0 = \langle\omega\rangle. \tag{27}$$

Here we will not consider the problem in a general form, but restrict ourselves to the types of pulses given in [29].

Single-cycle Gaussian pulse. It can be specified in the form (16), assuming that $q \to 0$. In this case, this expression will turn into the expression

$$g(x) = Cx\exp(-x^2/a^2). \tag{28}$$

The ratio of the maximum energy density at the collapse point to the total energy (10) takes the form

$$\frac{\epsilon_{\mathrm{m}}}{\mathcal{E}} = \frac{2}{3a^3}\left(\frac{2}{\pi}\right)^3. \tag{29}$$

Now we express the same value in terms of spectral parameters. Substituting the pulse spectrum

$$|G(\omega)|^2 = A^2\omega^2\exp\left(-\frac{\omega^2 a^2}{2c^2}\right)$$

into Eqns (13) and (14), we arrive at the relations

$$\frac{\omega_0}{c} = \frac{1}{a}\sqrt{\frac{8}{\pi}} \approx \frac{1.77}{a}, \quad \frac{\langle\omega^2\rangle}{c^2} = \frac{3}{a^2},$$
$$\frac{\sigma_\omega}{c} = \frac{1}{a}\sqrt{3-\frac{8}{\pi}} \approx \frac{0.67}{a}. \tag{30}$$

Here A is a constant related to the constant C from Eqn (28). Performing simple transformations, we obtain an analogue of expression (19):

$$\frac{\epsilon_{\mathrm{m}}}{\mathcal{E}} = \sqrt{\frac{8}{\pi}}\,\frac{2}{3}\left(3-\frac{8}{\pi}\right)^{-1/2}\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3\frac{\sigma_\omega}{\omega_0}$$
$$\approx 1.6\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3\frac{\sigma_\omega}{\omega_0}. \tag{31}$$

Half-cycle Gaussian pulse. It is also obtained by passing to the limit of zero modulation frequency in Eqn (16), but only if the sine is first replaced by the cosine:

$$g(x) = C\exp(-x^2/a^2). \tag{32}$$

Expression (10) takes the form

$$\frac{\epsilon_{\mathrm{m}}}{\mathcal{E}} = \frac{2}{3ea^3}\left(\frac{2}{\pi}\right)^{3/2}. \tag{33}$$

The spectrum of the pulse and its parameters are determined by the relations

$$|G(\omega)|^2 = A^2\exp\left(-\frac{\omega^2 a^2}{2c^2}\right), \quad \frac{\omega_0}{c} = \frac{1}{a}\sqrt{\frac{2}{\pi}} \approx \frac{0.8}{a},$$
$$\frac{\langle\omega^2\rangle}{c^2} = \frac{1}{a^2}, \quad \frac{\sigma_\omega}{c} = \frac{1}{a}\sqrt{3-\frac{2}{\pi}} \approx \frac{0.603}{a}, \tag{34}$$

and the reduced energy density is

$$\frac{\epsilon_{\mathrm{m}}}{\mathcal{E}} \approx 2.56\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3\frac{\sigma_\omega}{\omega_0}. \tag{35}$$

Half-cycle soliton-like pulse. For solitons (stable field configurations maintained by the spatial profile of the nonlinear addition to the refractive index of the medium), the field cannot decrease faster than exponentially with respect to the coordinate absolute value. The closest regular function corresponding to such a law is hyperbolic secant. In Ref. [9], a half-cycle soliton-like pulse is specified by the following field dependence:

$$g(x) = \frac{C}{\cosh^2(x/a)}. \tag{36}$$

Expression (10) for this field has the form

$$\frac{\epsilon_{\mathrm{m}}}{\mathcal{E}} = \frac{4}{27\pi a^3}. \tag{37}$$

The Fourier transform of function (36) is implemented by integration over the contour in the complex plane $-\infty \to +\infty \to +\infty + \mathrm{i}\pi a \to -\infty + \mathrm{i}\pi a \to -\infty$ and reduces to one residue (at the point $\mathrm{i}\pi a/2$). As a result, we have the relations

$$G(\omega) = \frac{A\omega}{\sinh(\pi\omega a/c)}, \tag{38}$$

$$|G(\omega)|^2 = \frac{A^2\omega^2}{\sinh^2(\pi\omega a/c)}, \quad \frac{\omega_0}{c} = \frac{18}{\pi^3 a}\zeta(3) \approx \frac{0.698}{a},$$
$$\frac{\langle\omega^2\rangle}{c^2} = \frac{4}{5a^2}, \quad \frac{\sigma_\omega}{c} = \frac{2}{a}\sqrt{\frac{1}{5}-\left(\frac{3}{\pi}\right)^4\left[\frac{\zeta(3)}{\pi}\right]^2} \approx \frac{0.56}{a}. \tag{39}$$

Here

$$\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$$

is the corresponding value of the Riemann zeta function. As a result, the ratio (37) is expressed in terms of the spectrum parameters as follows:

$$\frac{\epsilon_{\mathrm{m}}}{\mathcal{E}} \approx 2.72\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3\frac{\sigma_\omega}{\omega_0}. \tag{40}$$

Single-cycle soliton-like pulse. It corresponds to the derivative of the pulse (36), i.e.

$$g(x) = \frac{C\sinh(x/a)}{\cosh^3(x/a)}. \tag{41}$$

For this case, expression (10) has the form

$$\frac{\epsilon_{\mathrm{m}}}{\mathcal{E}} = \frac{5}{4\pi a^3}. \tag{42}$$

Since the derivative (41) is proportional to the derivative of the half-cycle pulse (36), its spectrum is obtained from the corresponding expression (39) by multiplying by ω^2:

$$|G(\omega)|^2 = \frac{A^2\omega^4}{\sinh^2(\pi\omega a/c)}. \tag{43}$$

To calculate the spectrum parameters, we will use the general formula for the integrals:

$$\int_0^\infty \frac{\omega^n\,\mathrm{d}\omega}{\sinh^2[\pi\omega a/(2c)]} = 4\frac{n!}{(\pi a)^{n+1}}\zeta(n), \tag{44}$$

also applicable to obtain results using the formulae (39). Thus, for the spectrum parameters (43) we obtain the relations

$$\frac{\omega_0}{c} = \frac{450}{\pi^5 a}\zeta(5) \approx \frac{1.525}{a},$$
$$\frac{\langle\omega^2\rangle}{c^2} = \frac{20}{7a^2}, \quad \frac{\sigma_\omega}{c} \approx \frac{0.729}{a}. \tag{45}$$

Finally, the expression for the reduced energy density takes the form

$$\frac{\epsilon_{\rm m}}{\mathcal{E}} \approx 2.7\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3 \frac{\sigma_\omega}{\omega_0}. \tag{46}$$

Half-cycle Lorentz pulse. Its generating function has the form of a resonance curve:

$$g(x) = \frac{C}{a^2 + x^2}. \tag{47}$$

The relation (10) has the form

$$\frac{\epsilon_{\rm m}}{\mathcal{E}} = \frac{9}{32\pi^2 a^3}. \tag{48}$$

The pulse spectrum (47) and its parameters are determined by the relations

$$|G(\omega)|^2 = A^2 \exp\left(-\frac{2a\omega}{z}\right), \quad \frac{\omega_0}{c} = \frac{1}{2a},$$
$$\frac{\langle\omega^2\rangle}{c^2} = \frac{1}{2a^2}, \quad \frac{\sigma_\omega}{c} = \frac{1}{2a}, \tag{49}$$

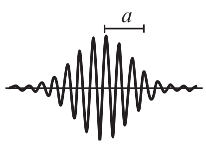
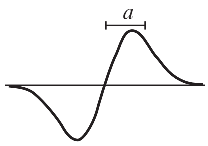
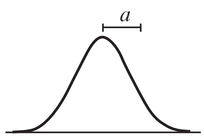
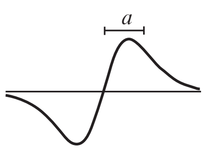
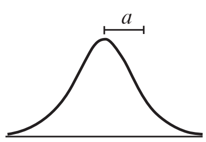
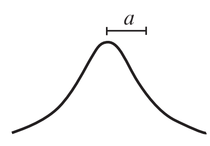
and the expression for the reduced energy density (48) is converted into the expression

$$\frac{\epsilon_{\rm m}}{\mathcal{E}} = \frac{9}{\sqrt{2\pi}}\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3 \frac{\sigma_\omega}{\omega_0} \approx 3.59\left(\frac{\sqrt{2\pi}}{\lambda_0}\right)^3 \frac{\sigma_\omega}{\omega_0}. \tag{50}$$

The considered characteristics of single-cycle and half-cycle nonparaxial collapsing pulses are summarised in Table 1.

Thus, formulae (31), (35), (40), (46) and (50) show that the expression for the limiting reduced density of electromagnetic energy of the collapsing pulse (19), found in Ref.

Table 1. Spectral parameters and coefficients at reduced energy density (10) for various types of pulses.

Pulse type	Shape	Pulse parameters		Coefficient of reduced energy density
		ω_0	σ_ω	
Gaussian quasi-monochromatic		qc	c/a	1 [see (19)]
Single-cycle Gaussian		$1.77c/a$	$0.67c/a$	0.6
Half-cycle Gaussian		$0.8c/a$	$0.603c/a$	0.96
Single-cycle soliton-like		$1.525c/a$	$0.729c/a$	1.013
Half-cycle soliton-like		$0.698c/a$	$0.56c/a$	1.02
Half-cycle Lorentzian		$0.5c/a$	$0.5c/a$	1.347

[8] for the quasi-monochromatic case, remains valid and for single-cycle and half-cycle pulses up to a factor of 0.6–1.4. The difference is due to the shape of the ultrashort pulse. The maximum value of the reduced energy density is obtained for a half-cycle Lorentz pulse. Note that, according to Ref. [29], the probability of atom ionisation in the field of single-cycle and half-cycle pulses also depends on their shape.

7. Conclusions

In this paper, by an example of the simplest exact solution of the Maxwell equations with finite field energy, we investigated the properties of electromagnetic beams that are beyond the paraxial and monochromatic approximations, which allowed considering single-cycle and half-cycle pulses.

It is shown that the reduced maximum density of the electromagnetic energy of a collapsing pulse, found earlier for a quasi-monochromatic Gaussian pulse, retains its value in cases of single-cycle and half-cycle pulses up to a factor of 0.6–1.4, depending on the shape of the pulse. It is found that the shape of an arbitrary pulse, as it propagates to the centre of collapse, is transformed into the shape of its derivative.

The investigated solutions of the equations for the electromagnetic field correspond to spherically symmetric solutions of the scalar wave equation. The latter also has exact analytical solutions without radial symmetry corresponding to spherical harmonics with index $l \neq 0$ (see Appendix). Obviously, the theory presented in this article can also be constructed for them.

The found relations between the maximum attainable density of electromagnetic energy and the spectrum of a collapsing pulse can be useful for heuristic estimates. For practical purposes, it is desirable to conduct a similar study of the parameters of electromagnetic pulses corresponding to solutions of the scalar wave equation with cylindrical symmetry.

In conclusion, we note that in a number of papers (see Section 2), the motion of charged particles in beams described by exact solutions of the Maxwell equations is considered. This is undoubtedly of certain interest in cases (including the present work), when the solutions contain an arbitrary function of time and allow simulating pulses of an arbitrary temporal shape with an arbitrary number of cycles. At the same time, the spatial structure of the field, as a rule, is quite concrete (in this work, it is spherical harmonic) and can be very far from that usually encountered in practice. Therefore, in [16–18, 22] optical systems were proposed for converting laser radiation into beams, which are described by exact solutions of the field equations. Another approach is to search for new exact solutions of Maxwell's equations, which more fully reproduce the characteristics of real laser radiation [19, 26, 30].

Acknowledgements. The authors are grateful to A.M. Fedotov, A.P. Kiselev, M.V. Perel', A.B. Plachenov, M.V. Fedorov, and P.V. Zinin for useful discussions.

This work was supported by Basic Funding under Theme No. 0023-0002-2018, by the Russian Foundation for Basic Research (Grant Nos 19-02-00394, 18-08-01066, 18-29-17039 and 17-08-01286), as well as by the Presidium of the Russian Academy of Sciences (PP RAS No. 7, Actual Problems of Photonics, Sensing of Inhomogeneous Media and Materials Scientific Research Programme).

Appendix

In addition to the spherically symmetric solution (5), the scalar wave equation (4) has a solution in the form of spherical harmonics:

$$u(\mathbf{r},t) = Y_{lm}(\theta,\varphi)R_l(r,t), \quad \text{(A1)}$$

where $Y_{lm}(\theta,\varphi)$ is a spherical function, and the radial function $R_l(r,t)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R_l}{\partial r}\right) - l(l+1)R_l = \frac{r^2}{c^2}\ddot{R}_l. \quad \text{(A2)}$$

We now express the function $R_l(r,t)$ in terms of an arbitrary smooth function $f(x)$, using the fact that for $l = 0$ the solution is representable in form (5). To do this, we repeat the reasoning given in [31]. Suppose we find a solution $R_l(r,t)$ of Eqn (A2) for some l. We write Eqn (A2) for $l = l + 1$:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{l+1}}{\partial r}\right) - (l+1)(l+2)R_{l+1} = \frac{r^2}{c^2}\ddot{R}_{l+1}. \quad \text{(A3)}$$

Let us substitute into (A3) the function

$$R_{l+1} = r^l\frac{\partial}{\partial r}\left(\frac{R_l}{r^l}\right). \quad \text{(A4)}$$

For the right-hand side of Eqn (A3), taking Eqn (A2) into account, we obtain the expression

$$\frac{r^2}{c^2}\ddot{R}_{l+1} = \frac{\partial}{\partial r}\left(\frac{r^2}{c^2}\ddot{R}_l\right) - \frac{l+2}{r}\left(\frac{r^2}{c^2}\ddot{R}_l\right) = r^2\frac{\partial^3 R_l}{\partial r^3}$$

$$-(l-2)r\frac{\partial^2 R_l}{\partial r^2} - (l+1)(l+2)\frac{\partial R_l}{\partial r} + l(l+1)(l+2)\frac{R_l}{r}, \quad \text{(A5)}$$

and for the left-hand side of Eqn (A3) we have the expression

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{l+1}}{\partial r}\right) - (l+1)(l+2)R_{l+1} = r^2\frac{\partial^3 R_l}{\partial r^3}$$

$$-(l-2)\frac{\partial^2 R_l}{\partial r^2} - (l+1)(l+2)\left(\frac{\partial R_l}{\partial r} - \frac{R_l}{r}\right). \quad \text{(A6)}$$

From (A5) and (A6) it follows that equality (A3) holds.

As a result, we have the recurrence formula (A4), which, together with formula (5), gives a solution to the radial wave equation for a spherical wave for any l containing an arbitrary smooth function $f(x)$.

Finally, we present the expressions for the radial functions $R_l(r,t)$ for the first three harmonics:

$$R_0(r,t) = \frac{f(ct+r) - f(ct-r)}{r},$$

$$R_1(r,t) = \frac{f'(ct+r) + f'(ct-r)}{r}$$

$$-\frac{f(ct+r) - f(ct-r)}{r^2}, \quad \text{(A7)}$$

$$R_2(r,t) = \frac{f''(ct+r) - f''(ct-r)}{r} - 3\frac{f'(ct+r) + f'(ct-r)}{r^2} + 3\frac{f(ct+r) - f(ct-r)}{r^3}.$$

References

1. Brabec T., Krausz F. *Rev. Mod. Phys.*, **72** (2), 545 (2000).
2. Kärtner F.X. (Ed.) *Few-Cycle Laser Pulse Generation and Its Applications* (Springer Science & Business Media, 2004) Vol. 95, Topics in Applied Physics.
3. Silva F. et al. *Opt. Lett.*, **43** (2), 337 (2018).
4. Akturk S. et al. *J. Opt.*, **12** (9), 093001 (2010).
5. Alonso B., Miranda M., Sola I.J., Crespo H. *Opt. Express*, **20** (16), 17880 (2012),
6. Pariente G., Gallet V., Borot A., Gobert O., Quéré F. *Nat. Photonics*, **10** (8), 547 (2016).
7. Alonso B., Perez-Vizcaino J., Minguez-Vega G., Sola I.J. *Opt. Express*, **26** (8), 10762 (2018).
8. Artyukov I.A., Vinogradov A.V., Dyachkov N.V., Feshchenko R.M. *Quantum Electron.*, **48** (11), 1073 (2018) [*Kvantovaya Elektron.*, **48** (11), 1073 (2018)].
9. Kiselev A.P. *Opt. Spectrosc.*, **102** (4), 603 (2007) [*Opt. Spektrosk.*, **102** (4), 661 (2007)].
10. Heyman E., Felson L.B. *J. Opt. Soc. Am. A*, **6**, 806 (1989).
11. Ziolkowski R.W. *Phys. Rev. A*, **39**, 2005 (1989).
12. Bateman H. *The Mathematical Analysis of Electrical and Optical Wave-motion on the Basis of Maxwell's Equations* (Cambridge: University Press, 1915).
13. Zangwill A. *Modern Electrodynamics* (Cambridge: University Press, 2012).
14. Tikhonov A.N., Samarskii A.A. *Uravneniya matematicheskoy fiziki* (Equations of Mathematical Physics) (Moscow: Nauka, 2004).
15. Isakovich M.A. *Obshchaya akustika* (General Acoustics) (Moscow: Nauka, 1973).
16. Hellwarth R.W., Nouchi P. *Phys. Rev. E*, **54**, 889 (1996).
17. Feng S., Winful H.G., Hellwarth R.W. *Opt. Lett.*, **23**, 385; 1141 (1998).
18. Feng S., Winful H.G., Hellwarth R.W. *Phys. Rev. E*, **59**, 4630 (1999).
19. Fedotov A.M., Korolev K.Yu., Legkov M.V. *Proc. SPIE*, **6726**, 672613 (2007).
20. Narozhnyi N.B., Fofanov M.S. *JETP*, **90**, 753 (2000) [*Zh. Eksp. Teor. Fiz.*, **117**, 867 (2000)].
21. Fedotov A.M. *Laser Phys.*, **19** (2), 214 (2009).
22. Gonoskov I., Aiello A., Heugel S., Leuchs G. *Phys. Rew. A*, **86**, 053836 (2012).
23. Dirac P.A.M. *Proc. R. Soc. London, Ser. A*, **5**, 148 (1938).
24. Wheeler J.A., Feynman R.P. *Rev. Mod. Phys.*, **15**, 157 (1945).
25. Lekner J. *Theory of Electromagnetic Pulses* (San Rafael, USA: Morgan & Claypool Publishers).
26. Kiselev A.P., Plachenov A.B., Chamorro-Posada P. *Phys. Rew. A*, **85**, 043835 (2012).
27. Gelfand I.M., Minlos R.A., Shapiro Z.Ya. *Representations of the Rotation Group and Lorentz Group and Their Applications* (Oxford: Pergamon press, 1963; Moscow: Fizmatlit, 1958).
28. Akhiezer A.I., Berestetskiy V.B. *Kvantovaya elektrodinamika* (Quantum Electrodynamics) (Moscow: Nauka, 1969).
29. Keldysh L.V. *Phys. Usp.*, **60**, 1187 (2017) [*Usp. Fiz. Nauk*, **187** (11), 1280 (2017)].
30. Fialkovsky I.V., Perel M.V., Plachenov A.B. *J. Math. Phys.*, **55**, 112902 (2014).
31. Landau L.D., Lifshitz E.M. *Quantum Mechanics* (London: Pergamon Press, 1958).