

Effect of elastic anisotropy on thermally induced distortions of a laser beam in single cubic syngony crystals with radial cooling. Part I

A.G. Vyatkin, E.A. Khazanov

Abstract. We have studied thermally induced distortions of a beam in single cubic syngony crystals of all symmetry groups with an anisotropic elastic stiffness tensor. The arithmetic mean and the difference between the thermally induced phase incursions of eigenpolarisations, as well as the angle of incidence of these polarisations, are calculated for a long rod and a thin disk with radial cooling under uniform volume pumping of the active element. The position of the specific crystal orientations unrelated to its symmetry elements is investigated. The effective values of the thermo-optical constant Q in two specific orientations, as well as the thermo-optical constant P in an arbitrary orientation, are found.

Keywords: high average power lasers, photoelastic effect, thermally induced depolarisation, thermo-optical constants, anisotropy of cubic crystals.

1. Introduction

Thermal effects are an important factor limiting the power and quality of the output radiation of solid-state lasers [1, 2]. The heat release in optical elements leads not only to an increase in the volume-average temperature, but also to the emergence of temperature gradients, which, in turn, are a source of elastic stresses. Due to the photoelastic effect in optical elements of any nature (glasses, single crystals of all symmetry groups, and ceramics), there appears an anisotropic and volume nonuniform addition to the permittivity tensor, causing in the general case both phase and polarisation distortions of the laser radiation passing through them.

Thermally induced anisotropy arises even in initially optically isotropic media (cubic single crystals, glasses, and ceramics). In these media, the process of transferring energy from the initial polarisation to the orthogonal one (in the sense of the scalar product), called depolarisation, has a dramatic effect on the optical quality of the transmitted beam [3, 4]. In media with natural birefringence (noncubic single crystals and oriented noncubic ceramics), the thermally induced change in the polarisation of transmitted radiation is usually insignificant, except for the cases of its propagation at small angles to the optical axis [2], and astigmatic phase distortion markedly affect the beam quality. Due to the anisot-

ropy of photoelastic properties of all crystals including cubic ones, the nature of thermally induced anisotropy substantially depends on the orientation of the crystallographic axes in the optical element [5, 6].

Thermally induced beam distortions in glasses were studied theoretically and experimentally in the 1960s and 1970s [7–11]. In the early 1970s, distortions were studied in cubic single crystals of the symmetry groups $m\bar{3}m$, 432 and $\bar{4}3m$, which for brevity we will call $m\bar{3}m$ crystals (the notation is explained in Section 2). Initially, the $[111]$ orientation was investigated ([3, 4, 12–15]), and a few years later, the $[001]$ orientation (see Refs [16, 17]) and the critical orientation [18, 19] were examined. In the 2000s, the method proposed in [20] was used to study thermally induced depolarisation in arbitrarily oriented $m\bar{3}m$ crystals [21, 22], and at the same time, special properties of the $[011]$ orientation in them (see Refs [22–24]) were pointed out. Thermally induced depolarisation in the remaining cubic single crystals – symmetry groups 23 and $m\bar{3}$ (for brevity, we will call them $m\bar{3}$ crystals, see Section 2) – was theoretically considered in [25]. The study of phase and polarisation distortions of the beam in cubic $m\bar{3}m$ ceramics began in the 2000s [26–32]. Recently, distortions in $m\bar{3}$ ceramics have been theoretically investigated [33–35].

In the vast majority of theoretical works, the elastic properties of cubic crystals (tensors of elastic stiffness and elastic compliance) were thought to be isotropic, although in the early years it was already known that this was only an approximation, and a solution to the elasticity problem was also found for the parabolic temperature profile [36, 37]. Anisotropy of elastic properties was taken into account in some works devoted to thermo-optics of both cubic and noncubic crystals, but only for the simplest orientations [38–41].

In the present work, thermally induced distortions of a beam in single cubic syngony crystals of all symmetry groups are considered taking into account the anisotropy of their elastic properties and for an arbitrary orientation of the crystallographic axes. In Section 2, we introduce notations necessary for the statement of the problem. Section 3 discusses the expressions used to describe the photoelastic effect in media with a nonuniform temperature distribution. Section 4 is devoted to a review of existing analytical solutions to the problems of heat conduction and elasticity and the choice of the most accurate solution for long rod geometry. In Section 5, we obtain expressions for thermally induced beam distortions in cubic crystals in the form of a long rod and a thin disk. The arithmetic mean and the difference between the phase incursions of eigenpolarisations, as well as the angle of incidence of eigenpolarisations are found; the specific crystal orientations

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are discussed; and effective values of thermo-optical constants P and Q are determined.

2. Statement of the problem and some notations

In the framework of this paper, we assume that the optical element has the shape of a cylinder, and its axis coincides with the z axis of the laboratory coordinate system (x, y, z) and with the propagation direction of probe radiation. The element can be cut from the bulk of the crystal in different directions. The orientation of the element is uniquely determined by the position of its axis in the crystallographic coordinate system (a, b, c) . This can be done using the first two of the three Euler angles, i.e. azimuthal (α) and polar (β) (Figs 1a, 1b, and 1d). As a result, the coordinate system (x'', y'', z) obtained from the crystallographic system due to rotation through these two angles will be bound with the optical element. In this case, the third Euler angle, Φ , will be equal, with the opposite sign, to the rotation angle of the element around its axis relative to the laboratory coordinate system (x, y, z) (Figs 1c and 1d). The pair of Euler angles (α, β) defines the crystal orientation, also denoted by Miller indices $[MNP]$. In this paper, significant attention is paid to orientations of form $[MON]$ and $[MMN]$ shown in Figs 1e and 1f. In addition, we will not distinguish between orientations obtained from each other by cyclic permutation of indices $[NPM]$, $[PMN]$ and changing the signs of indices $[MN\bar{P}]$, $[\bar{M}N\bar{P}]$, etc.) and consider them equivalent [25].

In addition to the Cartesian coordinate system, we will use cylindrical coordinates. Let the cylindrical coordinate system (r, φ, z) correspond to the laboratory Cartesian system

(x, y, z) , then the system (x'', y'', z) will correspond to the system (r, φ, z) , where

$$\varphi_{\Phi} = \varphi + \Phi. \quad (1)$$

Large heat release and large thermally induced distortions in laser systems are characteristic of active elements and Faraday cells. In this paper, we will consider media that do not rotate the polarisation plane; therefore, we will call for definiteness the optical element active in thermo-optics calculations.

We will also assume, unless explicitly stated otherwise, that the length of the active element L is much greater or much less than its radius R (a long rod or a thin disk, respectively), its heating is uniform along z , and cooling is carried out from the side surface. In these approximations, the temperature and elastic stresses in the disk, as well as the temperature in the rod, are independent of z , and the stresses and strains in the rod are almost constant at a distance of more than one or two radii from its ends, i.e., along most of the rod's length. Since thermally induced distortions of the beam accumulate during its propagation in the medium, we will neglect the end effects in the rod following the generally accepted approach, with the exception of one of the sections of the second part of the paper.

In this paper, as in previous ones, we consider lateral cooling of a thin disk. For this case, there is an exact and simple analytical solution to the problem of elasticity, the form of which is independent of the material parameters of the medium and the parameters of thermal contacts with the coolant or heat sink, and the distortions themselves are relatively large. It should be noted that radial cooling in the case of a thin disk leads to far from optimal temperature conditions. In the case of face cooling, the temperature differences turn out to be significantly lower and, in addition, the temperature gradients are often almost parallel to the z axis. Both these factors lead to a significant reduction in thermally induced distortions of the probe beam. For such a cooling regime in the case of an isotropic elastic stiffness tensor, several approximate solutions are known [32, 42, 43], one of which [32] requires weak heat removal, which contradicts the ideology of a high average power laser, and in the second, as well as in the solution from [42], radial heat flux is completely neglected. Solution [43] is built on the basis of some approximation of the radial flux and, as follows from [44] and our comparison with the results of numerical simulation by the code from [45], does not have high accuracy for all sets of input parameters. In turn, the proximity of the directions of temperature gradients and the propagation of radiation can result in a strong effect of small deviations of the real stress field from approximate solutions on the directions of the gradients and, as a consequence, on the magnitude of thermally induced distortions of the beam. In this regard, the applicability of these solutions for calculating thermo-optics, in contrast to the geometry with lateral cooling, depends on the parameters of the problem (aspect ratio, quality of thermal contacts, shape of the heating beam) and requires a separate detailed study, which is advisable to carry out for a specific task.

In this paper, we focused on obtaining a general qualitative dependence of the degree of depolarisation on orientation in the simplest geometry. We believe that the revealed distinctive features will be applicable in the case of face heat removal, despite the impossibility of a quantitative assess-

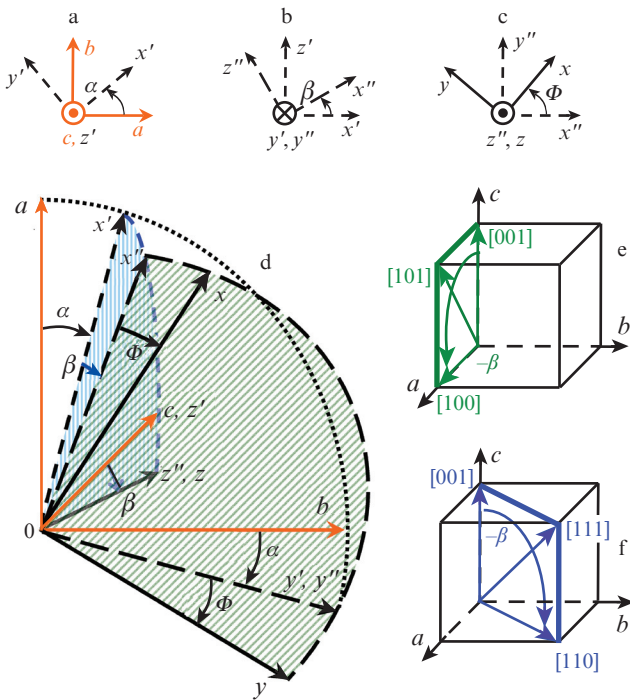


Figure 1. (Colour online) (a–d) Transition from crystallographic axes (a, b, c) through intermediate coordinate systems (x', y', z') and (x'', y'', z) to the laboratory coordinate system (x, y, z) by sequential rotation by the Euler angles (α, β, Φ) ; (e) $[MON]$ orientations ($\alpha = 0$) in the crystallographic coordinate system; (f) $[MMN]$ orientations ($\alpha = \pi/4$).

ment of the magnitude of thermally induced distortions of the beam.

There are two common forms of writing the expression for the photoelastic effect: in the form of a dependence of the dielectric impermeability tensor increment $\Delta \mathbf{B}$

$$\mathbf{B} = \boldsymbol{\varepsilon}^{-1} \quad (2)$$

on either the elastic strain tensor \mathbf{u} or on the elastic stress tensor $\boldsymbol{\sigma}$ [5]:

$$\Delta B_{u\,ij} = p_{ijkl} u_{kl}, \quad (3a)$$

$$\Delta B_{\sigma\,ij} = \pi_{ijkl} \sigma_{kl}, \quad (3b)$$

where $\boldsymbol{\varepsilon}$ is the permittivity tensor, and \mathbf{p} and $\boldsymbol{\pi}$ are the elasto-optic and piezo-optic material tensors of the fourth rank, respectively. The last two tensors are related via the elastic compliance tensor \mathbf{s} :

$$\pi_{ijkl} = p_{ijmn} s_{mkl}. \quad (4)$$

It is also possible to write the expression using the elastic stiffness tensor \mathbf{c} that is the inverse of the tensor \mathbf{s} :

$$p_{ijmn} = \pi_{ijkl} c_{klmn}, \quad (5)$$

$$s_{ijkl} c_{klmn} = \frac{1}{2} (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}).$$

In a cubic crystal, the fourth-rank tensors \mathbf{s} and \mathbf{c} are determined by three independent nonzero coefficients. Their appearance in the two-index Nye notation is shown schematically in Fig. 2a [5]. These crystals can be divided into two types. In the crystals of the symmetry groups $m\bar{3}m$, 432 and $\bar{4}3m$, which we have agreed to call $m\bar{3}m$ crystals, the photoelastic tensors \mathbf{p} and $\boldsymbol{\pi}$ have the same form. In the remaining cubic syngony crystals related to the symmetry groups $m\bar{3}$ and 23 , which we call $m\bar{3}$ crystals, these two tensors are determined by four independent coefficients, their general appearance being shown in Fig. 2b.

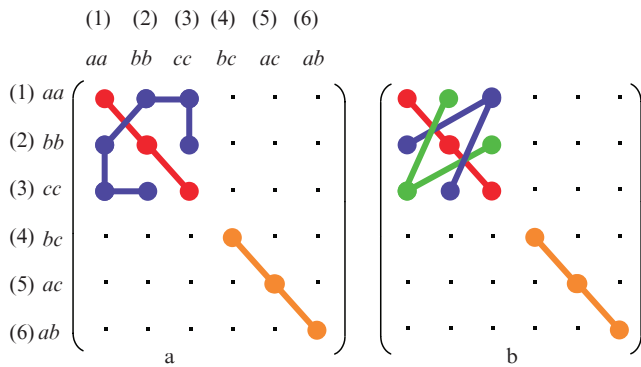


Figure 2. General appearance of material tensors of the fourth rank of cubic syngony crystals in the two-index Nye notation [5]: (a) piezo-optic ($\boldsymbol{\pi}$) and elasto-optic (\mathbf{p}) tensors of crystals of symmetry groups $m\bar{3}m$, 432 and $\bar{4}3m$, as well as elastic stiffness (\mathbf{c}) and compliance (\mathbf{s}) tensors of all symmetry groups; (b) tensors $\boldsymbol{\pi}$ and \mathbf{p} of crystals of symmetry groups $m\bar{3}$ and 23 . Identical elements are connected by straight lines.

3. Choice of the form of the photoelastic additive to the refractive index for the isotropic elasticity problem

The strain and stress tensors in (3) are related by the elasticity equation

$$u_{ij} - \alpha_{Tij}(T - T_0) = s_{ijkl} \sigma_{kl}, \quad (6)$$

where α_T is the thermal expansion tensor; T is the temperature field; and T_0 is the initial temperature at which an unstressed body is conditionally considered undeformed. It follows from equalities (3)–(6) that the increments $\Delta \mathbf{B}$ (3a) and (3b) cannot be made equal to each other in a medium of any symmetry even by selecting T_0 if the medium temperature depends on the coordinates:

$$\Delta \mathbf{B}_u - \Delta \mathbf{B}_\sigma = (T - T_0) \Delta \boldsymbol{\chi}_T, \quad (7)$$

$$\Delta \boldsymbol{\chi}_{T\,ij} = p_{ijkl} \alpha_{T\,kl}.$$

By virtue of (2), the tensor $\Delta \boldsymbol{\chi}_T$ in the linear approximation is, up to a factor, a correction to $\partial \boldsymbol{\varepsilon} / \partial T$ [46, 47]; therefore, in thermoelasticity problems it is important which formula from (3) is used. It can be shown that $\Delta \boldsymbol{\chi}_T$ has the same symmetry as the unperturbed ('cold') permittivity of the medium, $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}(T_0)$. In this regard, in the particular case of a cubic crystal, difference (7) between formulae (3) influences only the phase distortions of the beam, without affecting the polarisation [47]:

$$\Delta \boldsymbol{\chi}_{T\,ij} = \delta_{ij} \Delta \boldsymbol{\chi}_T, \quad (8a)$$

where

$$\Delta \boldsymbol{\chi}_T = (p_{aaaa} + p_{aabb} + p_{bbaa}) \alpha_T; \quad (8b)$$

δ_{ij} is the Kronecker symbol (note that the subsequent formulae in Ref. [47] are derived for $m\bar{3}m$ crystals and are not applicable to $m\bar{3}$ crystals).

The total increment of permittivity can be written as

$$\Delta \varepsilon_{ij}(\mathbf{u}, T) = (\partial \varepsilon_{ij} / \partial T)_u (T - T_0) + (\partial \varepsilon_{ij} / \partial u_{mn})_{T u_u} u_{mn}, \quad (9a)$$

$$\Delta \varepsilon_{ij}(\boldsymbol{\sigma}, T) = (\partial \varepsilon_{ij} / \partial T)_\sigma (T - T_0) + (\partial \varepsilon_{ij} / \partial \sigma_{mn})_{T \sigma_\sigma} \sigma_{mn}, \quad (9b)$$

where $kl \neq mn$. In this notation, the temperature derivatives of permittivity must be determined under different conditions: in (9a) at zero strains and in (9b) at zero stresses, and therefore their values are generally different, which is indicated by subscripts. In the usual notation of (2) and (3), expression (9) takes the form

$$\Delta \varepsilon_{ij}(\mathbf{u}, T) = (\partial \varepsilon_{ij} / \partial T)_u (T - T_0) - \varepsilon_{0ik} \varepsilon_{0lj} \Delta B_{ukl}, \quad (10a)$$

$$\Delta \varepsilon_{ij}(\boldsymbol{\sigma}, T) = (\partial \varepsilon_{ij} / \partial T)_\sigma (T - T_0) - \varepsilon_{0ik} \varepsilon_{0lj} \Delta B_{\sigma kl}. \quad (10b)$$

In a cubic crystal, substitution of (7) into (10) yields

$$(n_0^3/2) \Delta \boldsymbol{\chi}_T = \beta_u - \beta_\sigma, \quad (11)$$

$$\beta_\mu = \left(\frac{\partial n}{\partial T} \right)_\mu,$$

where n is the refractive index; and $n_0 = n(T_0)$ [46, 47].

It also follows from expressions (9) that the tensors \mathbf{p} and $\boldsymbol{\pi}$ must be isothermal, that is, they must be determined at a constant temperature [7, 46, 47] (adiabatic tensors determined at constant entropy are also used in a number of problems [5]).

In the light of these relations, we consider the thermo-optical constant P , a convenient characteristic for describing the average thermal lens between two eigenpolarisations (we will denote them by I and II) in cubic crystals:

$$[n_I(r) + n_{II}(r)]/2 = n(0) + [T(r) - T(0)]P. \quad (12)$$

This parameter was first introduced for long glass rods in [10]. In disk and rod geometries, P has a different form [16], and in addition, in the case of a thin disk, a term that is responsible for a change in the length of the active element is sometimes included [11, 16] (in Section 5.3.2, the quantity with an additional term is denoted by P^{tot}). Soms and Tarasov [16] also noted that the half-sum of the intrinsic refractive indices in cubic m3 crystals in the general case does not reduce to the form of (12) in the [011] orientation. Subsequently, it was found that in such crystals, P , in accordance with formula (12), can be introduced only for [001] and [111] orientations, as well as for statistical average refractive indices in optical ceramics, the obtained three values of P being different in each geometry [16, 30, 32] (m3m crystals were considered). In this regard, P is often determined in the [001] orientation in accordance with expression (12), and in other cases, additional terms are introduced into the formula (see [30, 32, 33, 35, 48]). Then the value of P depends only on the material parameters and geometry (a long rod or a thin disk) and, thus, with reservations as to the given geometry, is itself a material parameter. We will also stick to this approach, and the quantity introduced in accordance with (12) for an arbitrary orientation of the crystal is denoted by P^{eff} (see Section 5.3.2).

Consider an active element made of a cubic crystal with the [001] orientation ($\alpha = \beta = 0$) or of an isotropic material. In the former case, we additionally, following the tradition (see Introduction), consider elastic properties to be isotropic. The isotropic elastic compliance tensor in the two-index Nye notation has the following nonzero components in any Cartesian coordinate system [5] (see also Fig. 2a):

$$s_{11} = E^{-1}, s_{12} = -\nu s_{11}, s_{66} = 2(s_{11} - s_{12}), \quad (13)$$

where E and ν are Young's modulus and Poisson's ratio, respectively. Below, we will call for brevity such media elastically isotropic. Using the expressions for the stress tensor in cylindrical bodies from [49], it is easy to show that when using formula (10a), the expressions for P have the form

$$P_{\text{disk}}(\beta_u, \mathbf{p}) = \beta_u - (1 + 4\zeta_p)Q_{\text{disk}}, \quad (14a)$$

$$P_{\text{rod}}(\beta_u, \mathbf{p}) = \beta_u - (1 + 2\zeta_p)Q_{\text{rod}}, \quad (14b)$$

and when using formula (10b) in an equivalent form we have

$$P_{\text{disk}}(\beta_\sigma, \boldsymbol{\pi}) = \beta_\sigma + (1 + 2\zeta_\pi)Q_{\text{disk}}, \quad (15a)$$

$$P_{\text{rod}}(\beta_\sigma, \boldsymbol{\pi}) = \beta_\sigma + (1 + 4\zeta_\pi)Q_{\text{rod}}, \quad (15b)$$

where

$$Q_{\text{disk}} = \frac{1}{4}\alpha_T n_0^3 E \pi_s = \frac{1}{4}\alpha_T n_0^3 (1 + \nu) p_s; \quad (16)$$

$$Q_{\text{rod}} = Q_{\text{disk}}/(1 - \nu),$$

and

$$\begin{aligned} \zeta_p &= p_{12}^a/p_s; & \zeta_\pi &= \pi_{12}^a/\pi_s; \\ p_s &= p_{11} - p_{12}^a; & \pi_s &= \pi_{11} - \pi_{12}^a; \\ p_{12}^a &= \frac{1}{2}(p_{12} + p_{21}); & \pi_{12}^a &= \frac{1}{2}(\pi_{12} + \pi_{21}) \end{aligned} \quad (17)$$

(in m3m crystals $p_{12}^a = p_{12} = p_{21}$, $\pi_{12}^a = \pi_{12} = \pi_{21}$; see Fig. 2a). The quantity Q , like P , is also called the thermo-optical constant [10, 11]. Substituting (11) into (14), we can obtain an expression via a mixed set of variables:

$$P_{\text{disk}}(\beta_\sigma, \mathbf{p}) = \beta_\sigma + \frac{1}{4}\alpha_T n_0^3 [p_{11}(1 - \nu) + p_{12}^a(1 - 3\nu)], \quad (18)$$

$$P_{\text{rod}}(\beta_\sigma, \mathbf{p}) = \beta_\sigma + \frac{1}{4}\alpha_T n_0^3 [p_{11}(1 - 3\nu) + p_{12}^a(3 - 5\nu)].$$

As can be seen from these formulae, the form of the expression for the thermal lens substantially depends on the choice of variables. The literature review presented in Appendix 1, however, shows that there is no generally accepted agreement on the choice of a formula for describing the photoelastic effect, and the difference between them and the related issue of the condition for determining $\partial n/\partial T$ are usually ignored. In our opinion, from the sources mentioned in the review, the increment of the refractive index is correctly, unambiguously and fully described in [7, 9, 10, 46, 47, 50] and, with a slight caveat, in [4] (in a number of works, attention is paid only to birefringence and therefore they are not included in this list). The description is carried out by using one of two mathematically equivalent ways: using $(\partial n/\partial T)_u$ and (3a) [7, 46, 47, 50], and using $(\partial n/\partial T)_\sigma$ and (3b) [4, 7, 9, 10, 46, 47].

We consider the second option to be preferable for the following reasons:

1. The value of $(\partial n/\partial T)_\sigma$ is easier to measure than $(\partial n/\partial T)_u$, since for a uniformly heated body the condition for the absence of stresses is satisfied automatically [46, 50].

2. The components of the tensor $\boldsymbol{\pi}$ are also easier to measure than the components of the tensor \mathbf{p} , since it is easier to produce and measure a uniaxial load than a small uniaxial deformation.

3. As noted in [46], in the approximations of a long rod and a thin disk, the formulae for the components of the stress tensor [49] are simpler than those for the components of the strain tensor [32, 51].

4. In the case of the anisotropic elastic compliance tensor, this option is much more convenient, at least for cubic crystals (see Section 4.2).

On the contrary, the advantage of the first option is that the calculation of thermally induced distortions of the beam does not require for the knowledge of Young's modulus of the medium. However, this is not a very significant factor, since this parameter is usually well known. The second advantage is manifested in noncubic crystals, in which, due to the asymmetric part of the strain tensor, it is necessary to take into account the rotation of the optical axis or axes, for which the tensor \mathbf{p} , asymmetric over the last pair of indices, is usu-

ally introduced [52–54]. It is impossible to take this effect into account through the tensors $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$. This is a very specific problem that requires a separate consideration, since the asymmetric part of the strain tensor is not found in the currently known analytical solutions to the elasticity problem (see Section 4.2).

Thus, the thermally induced distortions of the beam (see Section 5) will be expressed through the piezo-optic tensor.

4. Solution to the problem of thermoelasticity in anisotropic media

4.1. Parabolic temperature distribution with anisotropic thermal conductivity

The parabolic temperature distribution is one of the simplest in the problems of thermoelasticity. It arises in cylindrical optical elements under conditions of isotropic heat conduction, homogeneous lateral cooling and heat release in the volume of the element. The first of these conditions is satisfied only for isotropic media and cubic crystals, and also, due to the absence of longitudinal heat fluxes, for uniaxial crystals in $[001]$ and $[00\bar{1}]$ orientations.

Sirotnin [36] showed that the same temperature distribution is typical of optical elements with an arbitrary form of the thermal conductivity tensor of the medium at a given temperature at the boundary, which corresponds to highly efficient cooling. It is convenient to introduce the dimensionless temperature τ :

$$T - T_0 = T_{\text{int}} \tau, \quad (19)$$

where $T_{\text{int}} = P_{\Sigma}/(4\pi L\kappa^a)$; P_{Σ} is the total heat dissipation power in the optical element; and T_0 is the temperature of the coolant or the heat sink on the thermal contact surface. In isotropic media, the thermal conductivity coefficient κ is used as κ^a . In anisotropic media, the former is a tensor of the second rank, and

$$\kappa^a = \frac{\text{Sp}(\boldsymbol{\kappa}_{\perp})}{2} = \frac{(\kappa_{xx} + \kappa_{yy})}{2}, \quad (20a)$$

where

$$\boldsymbol{\kappa}_{\perp} = \begin{pmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{pmatrix}. \quad (20b)$$

With uniform heat dissipation, dimensionless temperature has the form

$$\tau(u) = 1 - u, \quad u = \frac{r^2}{R^2}. \quad (21)$$

Thus, T_{int} has a sense of the temperature difference inside the optical element. In the particular case of the isotropic transverse thermal conductivity tensor $\boldsymbol{\kappa}_{\perp}$, taking place for uniaxial media with the $[001]$ orientation, cubic crystals, and isotropic bodies, under an arbitrary boundary condition the solution will take the form

$$\tau(u) = 1 + \tau_0 - u, \quad (22)$$

differing from (21) by the presence of a constant component $\tau_0 = 2\kappa/(RH) = 2/\text{Bi}$, inversely proportional to the Biot number (Bi), where H characterises the quality of the thermal contact of the element with the coolant or the heat sink and is called the heat transfer coefficient or surface conductivity.

From a practical point of view, for the applicability of solution (19)–(21) in the case of the anisotropic tensor $\boldsymbol{\kappa}_{\perp}$, it is necessary that $\text{Bi} \gg 1$, and as κ for an upper bound estimate we take the maximum eigenvalue of the tensor. This condition means a restriction from below on the value of H and is fulfilled in the case of sufficient cooling. Consider, for example, an active element of radius $R = 1$ cm made of a YAG crystal with good thermal conductivity ($\kappa = 10 \text{ W m}^{-1} \text{ K}^{-1}$). When cooled using a cold finger through good thermal paste or indium foil ($H = 2 \text{ W cm}^{-2} \text{ K}^{-1}$ [50]), we obtain $\tau_0 = 0.1$ ($\text{Bi} = 20$), and when cooled through indium solder ($H = 20 \text{ W cm}^{-2} \text{ K}^{-1}$ [55]), $\tau_0 = 0.01$ ($\text{Bi} = 200$).

Thus, in cylindrical optical elements, a parabolic temperature profile is established when heat is uniform in volume. In noncubic crystals, high quality of the thermal contact with a heat sink, which can be achieved by contact of solids, is also a necessary requirement.

4.2. Review of analytical solutions to the problem of thermoelasticity

Sirotnin [38] and Parfenov et al. [39] obtained solutions to the elasticity problem for a long rod and a thin disk with a parabolic temperature profile and arbitrary symmetry and orientation of the crystal. The found stress fields have similarities with the fields observed in elastically isotropic [see (13)] media. These solutions are based on the fact that the equations of mechanical equilibrium due to their independence from the longitudinal coordinate z in these geometries are reduced to a two-dimensional form:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad (23a)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$

Such a system of equations with boundary conditions of the free boundary

$$(\sigma_{xx} \cos \varphi + \sigma_{xy} \sin \varphi)|_{x^2 + y^2 = R^2} = 0, \quad (23b)$$

$$(\sigma_{xy} \cos \varphi + \sigma_{yy} \sin \varphi)|_{x^2 + y^2 = R^2} = 0$$

is automatically satisfied by axially symmetric formulae for the in-plane components of the stress tensor (σ_{ij} for $i, j \neq z$)

$$\sigma_{rr} = A(u - 1),$$

$$\sigma_{\varphi\varphi} = A(3u - 1), \quad (24a)$$

$$\sigma_{r\varphi} = 0,$$

which in the Cartesian coordinate system have the form

$$\sigma_{xx} = A(x^2 + 3y^2 - R^2)/R^2,$$

$$\sigma_{yy} = A(3x^2 + y^2 - R^2)/R^2, \quad (24b)$$

$$\sigma_{xy} = -2Axy/R^2$$

and differ from the solution of the isotropic problem by an arbitrary coefficient A .

In a thin disk, out-of-plane stresses σ_{jz} are considered equal to zero. The solution to the elasticity problem for an arbitrary elastic compliance tensor s was first published without derivation in [36]. It can be obtained, for example, by the methods described in paper [39] devoted to the long rod geometry using the equations presented there. Recently, this solution was used for the simplest orientations of a tetragonal crystal in [40, 41], but in a more cumbersome form, through the elastic stiffness tensor.

The coefficient A for the disk, which we will denote by A_{disk} , is determined from a generalisation (6) of Hooke's law and the Saint-Venant deformation compatibility equation

$$\frac{\partial^2 u_{xx}}{\partial y^2} + \frac{\partial^2 u_{yy}}{\partial x^2} - 2\frac{\partial^2 u_{xy}}{\partial x \partial y} = 0, \quad (25)$$

reduced in the case of a parabolic temperature profile to a linear algebraic equation. We rewrite the result from [36] in the notation convenient for us:

$$A_{\text{disk}} = T_{\text{int}} \alpha_T^a a_{\text{disk}}/4, \quad \alpha_T^a = (\alpha_{T,xx} + \alpha_{T,yy})/2, \quad (26)$$

$$a_{\text{disk}} = 8(3s_{xxxx} + 3s_{yyyy} + 2s_{xyxy} + 4s_{xyxy})^{-1}.$$

Thus, in a thin disk of an arbitrary crystal with a parabolic temperature distribution, the stress field is axially symmetric. Due to the anisotropy of the material equation (6), the deformation field (as well as the displacement field) does not have in the general case this symmetry [40, 41].

In the geometry of a long rod, a solution to the elasticity problem for a parabolic radial temperature profile was obtained previously by two different methods. The solution [37, 39], which we will later call Parfenov's solution, is constructed as follows. First, we consider the problem $(\mathbf{u}_0, \boldsymbol{\sigma}_0)$ of a rod with fixed ends (out-of-plane deformations $u_{0jz} = 0$). It is proposed to divide relations (6) into two subsystems, i.e. for in-plane and out-of-plane deformations, respectively:

$$\begin{pmatrix} u_{0,xx} \\ u_{0,yy} \\ 2u_{0,xy} \end{pmatrix} = \begin{pmatrix} s'_{11} & s'_{12} & s'_{16} \\ s'_{12} & s'_{22} & s'_{26} \\ s'_{16} & s'_{26} & s'_{66} \end{pmatrix} \begin{pmatrix} \sigma_{0,xx} \\ \sigma_{0,yy} \\ \sigma_{0,xy} \end{pmatrix} + \begin{pmatrix} s'_{13} & s'_{14} & s'_{15} \\ s'_{23} & s'_{24} & s'_{25} \\ s'_{36} & s'_{46} & s'_{56} \end{pmatrix} \begin{pmatrix} \sigma_{0,zz} \\ \sigma_{0,yz} \\ \sigma_{0,xz} \end{pmatrix} + \begin{pmatrix} \alpha_{T,xx} \\ \alpha_{T,yy} \\ 2\alpha_{T,xy} \end{pmatrix} (T - T_0), \quad (27a)$$

$$\begin{pmatrix} u_{0,zz} \\ 2u_{0,yz} \\ 2u_{0,xz} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s'_{33} & s'_{34} & s'_{35} \\ s'_{34} & s'_{44} & s'_{45} \\ s'_{35} & s'_{45} & s'_{55} \end{pmatrix} \begin{pmatrix} \sigma_{0,zz} \\ \sigma_{0,yz} \\ \sigma_{0,xz} \end{pmatrix} + \begin{pmatrix} s'_{13} & s'_{23} & s'_{36} \\ s'_{14} & s'_{24} & s'_{46} \\ s'_{15} & s'_{25} & s'_{56} \end{pmatrix} \begin{pmatrix} \sigma_{0,xx} \\ \sigma_{0,yy} \\ \sigma_{0,xy} \end{pmatrix} + \begin{pmatrix} \alpha_{T,zz} \\ 2\alpha_{T,yz} \\ 2\alpha_{T,xz} \end{pmatrix} (T - T_0), \quad (27b)$$

where the prime indicates that the two-index Nye notation was introduced relative to the laboratory coordinate system, rather than to the crystallographic one, as in Fig. 2a. Then, it

is proposed to exclude out-of-plane stresses from (27), and then using (24b) and (25) we obtain again a linear algebraic equation for $A = A_{\text{rod}}$. Having solved it, we determine the in-plane stresses, after substituting which into (27b) we also find out-of-plane stresses.

After that, an approximate correction of the solution is made for the case of a rod that can expand uniformly over the cross section in the direction of the z axis: Corrections $\delta\sigma_{jz} = \sigma_{jz} - \sigma_{0jz}$ that are constant over the cross section of the rod are introduced, such that the cross section averages from σ_{jz} are equal to zero. This condition means a change in the boundary condition at the ends of the rod, namely the equality of the total forces applied to them to zero. The introduced corrections do not affect the differential equilibrium equations inside the rod, but lead to the recalculation of the constant component of the tensor \mathbf{u} , because in accordance with (6)

$$u_{ij} - u_{0ij} = s_{ijk}(\sigma_{kz} - \sigma_{0kz}). \quad (28)$$

The resulting solution is approximate, because at the ends of the rod the condition of a mechanically free boundary is not satisfied at individual points of the cross section. Due to the fact that the system of forces applied to each end face is balanced, in accordance with the Saint-Venant principle, the perturbation of the solution introduced by such a substitution of the boundary condition is concentrated in a limited region (on the order of the rod diameter) [49].

Due to the linearity of the thermoelasticity problem, by analogy with a thin disk, it is convenient to introduce the notation

$$A_{\text{rod}} = T_{\text{int}} \tilde{a}_p/4, \quad (29)$$

$$\sigma_{zz \text{ rod}} = \frac{1}{2} T_{\text{int}} [\tilde{d}_{p1}(2u - 1) + \tilde{d}_{p2}u \cos 2\varphi + \tilde{d}_{p3}u \sin 2\varphi]/4$$

and similar notations for σ_{xz} and σ_{yz} .

Note that in papers [37, 39] the thermal expansion tensor was incorrectly translated into Nye's notation, which is why these works lack coefficients 2 in the temperature terms in (27). The transformation rules for the tensors \mathbf{u} and $\boldsymbol{\alpha}_T$ must be the same so that equation (6) is fulfilled simultaneously in both forms. However, in the special cases considered in [37, 39], the corresponding components of the tensor $\boldsymbol{\alpha}_T$ were equal to zero; therefore, the error did not affect the final results.

Parfenov's solution has a significant limitation. It does not guarantee the fulfilment of the third equilibrium equation

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0, \quad (30a)$$

the third boundary condition on the side surface of the rod

$$(\sigma_{xz} \cos \varphi + \sigma_{yz} \sin \varphi)|_{x^2 + y^2 = R^2} = 0 \quad (30b)$$

and two Saint-Venant equations obtained from (25) by a cyclic change of the coordinates (x, y, z) . If the latter two equations are not satisfied in the solution for a thin disk as well, which can be attributed to the approximate nature of both solutions, then the noncompliance of (30) specific to Parfenov's solution requires special attention.

Another solution to the elasticity problem in a long rod is given in [36]; we will call it Sirotin's solution. The stress func-

tion method of Lehnitsky was used [56]. The solution is obtained in the form

$$\begin{aligned} A_{\text{rod}} &= T_{\text{int}} \tilde{a}_s / 4, \\ \sigma_{xz \text{ rod}} &= \frac{1}{2} T_{\text{int}} [\tilde{c}_s (2u - 1) - \tilde{c}_s u \cos 2\varphi - \tilde{b}_s u \sin 2\varphi] / 4, \\ \sigma_{yz \text{ rod}} &= \frac{1}{2} T_{\text{int}} [\tilde{b}_s (2u - 1) + \tilde{b}_s u \cos 2\varphi - \tilde{c}_s u \sin 2\varphi] / 4, \\ \sigma_{zz \text{ rod}} &= \frac{1}{2} T_{\text{int}} [\tilde{d}_{s1} (2u - 1) + \tilde{d}_{s2} u \cos 2\varphi + \tilde{d}_{s3} u \sin 2\varphi] / 4, \end{aligned} \quad (31)$$

where the expressions for the coefficients $\tilde{a}_s, \tilde{b}_s, \tilde{c}_s,$ and \tilde{d}_{sj} are given in Appendix 2. It is easy to show that this solution satisfies Eqns (30) but does not guarantee the axial symmetry of the out-of-plane shear deformations u_{rz} and $u_{\varphi z}$, which are constant in Parfenov's solution in accordance with (27b) and (28), and the fulfilment of all three compatibility conditions, including (25). It can be shown that u_{zz} is constant in the cross section of the rod in both solutions.

Thus, in both solutions for a long rod of an arbitrary crystal with a parabolic temperature distribution, only the in-plane part of the stress field and the deformation u_{zz} constant across the cross section are axially symmetric. In Parfenov's solution, the remaining out-of-plane deformations are constant over the cross section and, therefore, axially symmetric. Other components of the solutions do not possess axial symmetry in the general case. In addition, it is worth noting that in the general case, all three out-of-plane deformations in a rod with loose ends are nonzero, and so the use of the term 'plane strain' as applied to this case for crystals of arbitrary symmetry makes sense only after additional justification.

A quick analysis of the above-considered solutions to the elasticity problem suggests that the description of the photoelastic effect in the form of $(\partial n / \partial T)_\sigma$ and (3b) is more convenient than in the form of $(\partial n / \partial T)_u$ and (3a), not only in elastically isotropic media, but, at least, in cubic crystals, for which a theory based on an axially symmetric solution to the problem of elasticity is well developed (see references in the Introduction). Thus, in a thin disk, the stress field differs from the elastically isotropic case previously studied in detail [22, 25] only by the scalar coefficient, which, as shown in Section 5.1, in the calculation of birefringence only renormalises the heat release power. In a long rod, in addition to a similar renormalisation, it is needed to take into account the changes in the stress σ_{zz} , as well as the nonzero σ_{xz} and σ_{yz} . However, this procedure is simpler than substituting into (3a) the non-axially symmetric (in the general case) strain tensor found from the stress tensor and (6).

In the case of noncubic media, it is necessary to take into account the remark on the asymmetric part of the strain tensor given at the end of Section 3.

4.3. Solution to the elasticity problem for a cubic crystal

Let us consider the stress fields described in Section 4.2 in long rod and thin disk cylindrical elements made of cubic crystals. Because their elastic compliance tensor (Fig. 2a) is generally nonisotropic, the stresses will depend on the orientation of the crystallographic axes.

4.3.1 Solution for a thin disk

In the geometry of a thin disk with an arbitrary orientation $[MNP]$, also denoted by Euler angles (α, β) (see Section 2 and Fig. 1), the coefficient a_{disk} (26), which determines the stress tensor, can be written as

$$\begin{aligned} a_{\text{disk}} &= E_{[001]} / Z_{\text{disk}}, \\ Z_{\text{disk}} &= 1 + \frac{(1 + \nu_{[001]})(1 - \xi_s)(3f_1 - 1)}{4}, \end{aligned} \quad (32)$$

$$\begin{aligned} f_1 &= -\frac{\sin^2 2\beta + \sin^2 2\alpha \sin^4 \beta}{4} \\ &= -\frac{M^2 N^2 + M^2 P^2 + N^2 P^2}{(M^2 + N^2 + P^2)^2}, \end{aligned}$$

where

$$\xi_s = \frac{s_{66}}{2(s_{11} - s_{12})} = \frac{2s_{abab}}{s_{aaaa} - s_{aabb}} \quad (33)$$

is the elastic anisotropy parameter introduced, for example, in [6, 18]. In media with the anisotropic tensor s , by analogy with isotropic media, it is customary to use Young's modulus and Poisson's ratio, which in this case are functions of direction (Poisson's ratio can also be introduced as a function of two orthogonal directions) [6, 57, 58]. In formula (32), these parameters are used in the $[001]$ direction:

$$E_{[001]} = s_{11}^{-1}, \quad \nu_{[001]} = -s_{12}/s_{11}. \quad (34)$$

In an isotropic medium, (13) holds; therefore, $\xi_s = 1$ and

$$a_{\text{disk}}^{\text{iso}} = E. \quad (35)$$

Thus, under the assumption that the elastic compliance tensor of an isotropic medium differs from the anisotropic one only by the value of s_{66} ,

$$Z_{\text{disk}} = \frac{a_{\text{disk}}^{\text{iso}}}{a_{\text{disk}}}. \quad (36)$$

For the $[MNP]$ direction in the crystal, Young's modulus has the form [6, 58]

$$E_{[MNP]} = E_{[001]} [1 + 2(1 + \nu_{[001]})(1 - \xi_s)f_1]^{-1} \quad (37)$$

and does not coincide with a_{disk} . It should also be noted that the positive definiteness of the tensor s imposes the following restrictions on the elastic properties of crystals [6]:

$$E_{[001]} \geq 0, \quad -1 \leq \nu_{[001]} \leq 1/2, \quad \xi_s \geq 0. \quad (38)$$

In the simplest orientations $[001]$, $[011]$ and $[111]$ of the crystal, we have

$$f_1 [001] = 0, \quad f_1 [011] = -1/4, \quad f_1 [111] = -1/3. \quad (39)$$

Orientations $[001]$ and $[111]$ are the points of a global maximum and minimum of f_1 and Z_{disk} .

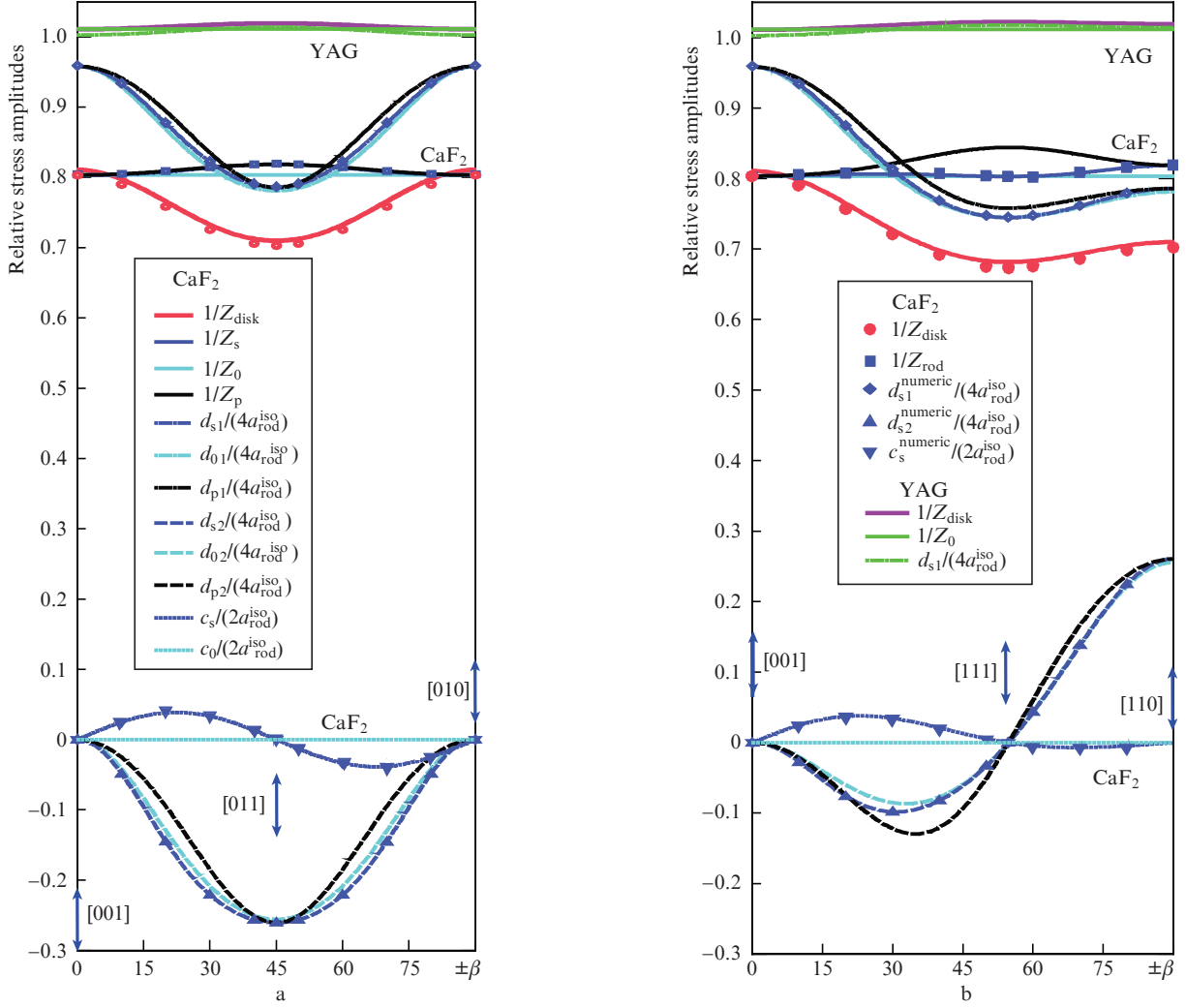


Figure 3. (Colour online) Analytically (curves) and numerically (points) calculated stresses for CaF_2 as functions of the Euler angle β in (a) $[M0N]$ (Euler angle $\alpha = 0$) and (b) $[MMN]$ ($\alpha = \pi/4$) orientations. For a thin disk: (red curve, \circ) $1/Z_{\text{disk}}$. For a long rod: (dark blue curves) complete and (light blue curves) simplified Sirotin's solutions, (black curves) Parfenov's solution; $1/Z_{s0p}$ (solid curves, \square), $d_{s101p1}/(4a_{\text{rod}}^{\text{iso}})^{-1}$ (dash and dot curves, \diamond), $\bar{d}_{s202p2}/(4a_{\text{rod}}^{\text{iso}})$ (dashed curves, \triangle) and $\bar{c}_{s0}/(2a_{\text{rod}}^{\text{iso}})$ (dotted curves, ∇). For comparison, the upper part of the figure shows the curves for YAG: (solid crimson curve) $1/Z_{\text{disk}}$, (solid green curve) $1/Z_s$ and (dash and dot green curve) $d_{s1}/(4a_{\text{rod}}^{\text{iso}})$.

Figure 3 shows the dependences of value of $1/Z_{\text{disk}}$ on the orientation of the crystallographic axes in the active element made of CaF_2 with significant elastic anisotropy and of YAG with weak elastic anisotropy (the ξ_s parameters are given in Table 1). The $1/Z_{\text{disk}}$ variation in CaF_2 is approximately 13%, while in YAG it is negligible.

To calculate the thermal lens, we also need the normal longitudinal deformation u_{zz} . Calculation by formula (6) yields

$$u_{zz} = \alpha_T(T - T_0) + (s_{xxzz} + s_{yyzz})(\sigma_{rr} + \sigma_{\varphi\varphi})/2 + [(s_{xxzz} - s_{yyzz})\cos 2\varphi + 2s_{xyzz}\sin 2\varphi](\sigma_{rr} - \sigma_{\varphi\varphi})/2 \quad (40)$$

and then

$$u_{zz} = \alpha_T T_{\text{int}} \left\{ (1 - u) + \frac{\nu_{[001]}}{Z_{\text{disk}}} \left(\frac{1}{2} - u \right) + \frac{1 + \nu_{[001]}}{Z_{\text{disk}}} \left[F_{1s} \left(\frac{1}{2} - u \right) + F_{2s} \frac{u}{2} \cos 2\varphi_\Phi + F_{3s} \frac{u}{2} \sin 2\varphi_\Phi \right] \right\}, \quad (41)$$

where the expressions for F_{is} , $i = 1, \dots, 3$ are given in Appendix 3, and φ_Φ is defined in (1).

4.3.2. Solution for a long rod

In a cubic crystal, the stress tensor is proportional to the scalar coefficient of thermal expansion; therefore, for the rod it is convenient to introduce into expressions (29) and (31) the coefficients

$$\{a_p, d_{pi}, a_s, b_s, c_s, d_{si}\} = \alpha_T^{-1} \{\tilde{a}_p, \tilde{d}_{pi}, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_{si}\}, \quad (42)$$

independent of it ($i = 1, 3$). Parfenov's solution is implicit and therefore not amenable to simple analysis. Sirotin's solution (A2.1)–(A2.3), which is very cumbersome in its general form, can be significantly simplified in a number of special cases. For the convenience of the further discussion, we will consider solutions to the elasticity problem (29) and (31) not in the laboratory coordinate system (x, y, z) , but in the system (x'', y'', z) bound to the active element; rotation through the angle Φ will be taken into account separately. The quantities

a_p , d_{p1} , a_s , and d_{s1} are the same in both coordinate systems, and the remaining coefficients (42) are denoted as

$$\{\bar{d}_{pi}, \bar{b}_s, \bar{c}_s, \bar{d}_{si}\} = \{d_{pi}, b_s, c_s, d_{si}\} \Big|_{\phi=0}. \quad (43)$$

Sirotin [36], in addition to the general solution, obtained simplified formulae for some simple orientations of crystals of various symmetries. For cubic crystals, such orientations are [001], [011], and [111], and the corresponding solutions are given in (A2.6). In the approximation of the isotropic problem of elasticity, both the general solution and expressions (A2.6) for all three orientations are reduced to the form

$$\begin{aligned} a_{\text{rod}}^{\text{iso}} &= E/(1-\nu) = (s_{11} + s_{12})^{-1}, \\ d_{\text{rod}1}^{\text{iso}} &= 4a_{\text{rod}}^{\text{iso}}, \\ b_{\text{rod}}^{\text{iso}} &= c_{\text{rod}}^{\text{iso}} = d_{\text{rod}2}^{\text{iso}} = d_{\text{rod}3}^{\text{iso}} = 0. \end{aligned} \quad (44)$$

For further consideration, it is convenient to introduce coefficients by analogy with the disk

$$\begin{aligned} Z_p &= a_{\text{rod}}^{\text{iso}}/a_p, \\ Z_s &= a_{\text{rod}}^{\text{iso}}/a_s, \\ \{D_1, D_2, D_3, D_b, D_c\} &= \{d_{s1}, \bar{d}_{s2}, \bar{d}_{s3}, \bar{b}_s, \bar{c}_s\}/(4a_s). \end{aligned} \quad (45)$$

The values of $1/Z_{sp}$ are the magnitudes of the in-plane stresses in the rod of arbitrary orientation in Sirotin's and Parfenov's solutions, normalised to the magnitude of the in-plane stresses in the elastically isotropic rod, which differs from the anisotropic one by the value of s_{66} [see (13), (33), and (34)]. The quantities D_i ($i = 1, 2, 3, b, c$) are the magnitudes of the various terms of the component σ_{zz} ($i = 1, 2, 3$), as well as of the components $\sigma_{x'z}$ and $\sigma_{y'z}$ ($i = b, c$) in Sirotin's solution normalised to the magnitude of in-plane stresses in the same orientation. The quantities $D_i/Z_s = \{\bar{d}_{si}, \bar{b}_s, \bar{c}_s\}/(4a_{\text{rod}}^{\text{iso}})$ are the magnitudes of the same stress components, normalised to the magnitude of the in-plane stresses in the elastically isotropic rod.

As in the case of a thin disk, in both solutions for a long rod, the in-plane stresses differ from elastically isotropic ones by the factors $1/Z_{sp}$ (Fig. 3), which, like for the disk, depend on the orientation of the crystal, but much weaker, and especially weak, in Sirotin's solution. In addition, out-of-plane stresses must be calculated using formulae other than those for the elastically isotropic case.

The strain tensor in a long rod made of a cubic crystal has the following properties: In Parfenov's solution, the strains $u_{rz} = u_{\phi z} = 0$, and u_{zz} is also constant over the cross section,

$$u_{zz} = \alpha_T T_{\text{int}}/2, \quad (46)$$

that is, such a state of the rod in these media can be considered as generalised plane-strained, as in the case of isotropic elasticity. In Sirotin's solution, the strain u_{zz} is the same, and u_{rz} and $u_{\phi z}$ are nonzero and depend on the coordinates in the cross section of the active element.

We studied out-of-plane shear strains and out-of-plane shear stresses in Sirotin's solution in more detail. The maximum value of the out-of-plane off-diagonal component of an arbitrary symmetric tensor X of the second rank at a given point is

$$X_{\perp} = \max_{\mu \perp z}(X_{\mu z}) = \sqrt{X_{rz}^2 + X_{\phi z}^2}, \quad (47)$$

and the maximum value of the diagonal in-plane component is

$$X_{\parallel} = \max_{\mu \perp z}(X_{\mu\mu}) = \frac{1}{2} \left[|X_{rr} + X_{\phi\phi}| + \sqrt{(X_{rr} - X_{\phi\phi})^2 + 4X_{r\phi}^2} \right]. \quad (48)$$

We compared with each other the normalised values of σ_{\perp} and u_{\perp} , maximum along the cross section of the optical element made of various cubic materials (see Table 1) of different orientations (note that the stress components reach a maximum on the cylinder generatrix):

$$\begin{aligned} M_1 &= \max_S(\sigma_{\perp})/\max_S(\sigma_{\parallel}) = \sqrt{b_s^2 + c_s^2}/(2a_s), \\ M_2 &= \max_S(\sigma_{\perp})/\max_S(\sigma_{zz}) = 2\sqrt{b_s^2 + c_s^2}/(d_{s1} + \sqrt{d_{s2}^2 + d_{s3}^2}), \\ N_1 &= \max_S(u_{\perp})/\max_S(u_{\parallel}), \\ N_2 &= \max_S(u_{\perp})/\max_S(u_{zz}). \end{aligned} \quad (49)$$

Table 1. Material properties of real and model m3m crystals used in the calculations.

Medium	ξ_{π}	ξ_s	$\nu_{[001]}$
CaF ₂	-0.47 [59, 60]	1.77 [61]	0.212 [61]
SrF ₂	-0.284 [60]	1.25 [62]	0.266 [62]
KCl	-1.2 [16]	2.69 [62]	0.135 [62]
YAG	3.2 [2]	0.965 [62]	0.25 [62]
1	3.2	1.77	0.212
2	3.2	0.44	0.212

In our calculations, M_i was a few percent, while N_i was in the range of 7%–15%, exceeding M_i by 3–6 times. From this we concluded that, contrary to our intuitive assumptions, the solution to the elasticity problem in an anisotropic rod is closer to the plane-stressed state characteristic of a thin disk than to the plane-strained state, at least in cubic crystals. Thus, in our opinion, it is impossible to consider Sirotin's solution to a generalisation of plane deformation.

4.3.3. Simplified Sirotin's solution for a rod

An important particular case of Sirotin's solution is the case when ξ_s differs slightly from unity. Analysis of the solution showed that for

$$\frac{1}{2} < \xi_s < 2, \quad 0 < \nu_{[001]} < 0.4 \quad (50)$$

a_s is virtually independent of orientation, and \bar{b}_s and \bar{c}_s are small (i.e., $\sigma_{x'z}$ and $\sigma_{y'z}$ are small; some of the dependences are shown in Fig. 3). Neglecting these small corrections, we can simplify formulae (A2.3) [taking into account (42)] to the form

$$\{d_{s1}, \bar{d}_{s2}, \bar{d}_{s3}\} \approx \{d_{01}, \bar{d}_{02}, \bar{d}_{03}\} = 4a_0 \{D_{01}, D_{02}, D_{03}\},$$

$$\{\bar{b}_s, \bar{c}_s\} \approx \{\bar{b}_0, \bar{c}_0\} = 0,$$

$$D_{01} = \frac{(s_{11}a_0)^{-1} + \nu_{[001]} + (1 + \nu_{[001]})F_{1s}}{1 + 2(1 + \nu_{[001]})F_{1s}}, \quad (51)$$

$$D_{02} = -D_0 F_{2s},$$

$$D_{03} = -D_0 F_{3s},$$

$$D_0 = \frac{1 + \nu_{[001]}}{1 + 2(1 + \nu_{[001]}) F_{1s}},$$

suitable for analytical analysis (see notation in Appendix 3), where as a_0 one can use any of the expressions for a_s from (A2.6), for example, based on the proximity of the studied orientation to the simplest ones. In the present paper, for definiteness, we will use the expression for $a_{s[001]}$. For the convenience of further analysis, we introduce, by analogy with the full solutions, the factor

$$Z_0 = a_{rod}^{iso}/a_0, \quad (52)$$

which is constant in this approximation. In the approximation of the isotropic elasticity problem, $Z_0 = D_{01} = 1$ and $D_{02} = D_{03} = 0$.

4.3.4. Numerical verification of analytical solutions to the elasticity problem

To verify the analytical solutions to the elasticity problem considered above, in the particular case of a cubic crystal, we calculated the stress fields in a long rod and a thin disk using a three-dimensional commercial FEM code for different ξ_s and $\nu_{[001]}$. The aspect ratio $L/2R$ was increased in the rod and decreased in the disk until the numerical solution began to coincide with one of the analytical ones. The obtained dependences of some stresses in YAG and CaF₂ in the middle cross section along the longitudinal coordinate z for $L/2R = 5$ and $L/2R = 1/40$ are shown in Fig. 3 as functions of the Euler angle β for the orientations $[M0N]$ ($\alpha = 0$) and $[MMN]$ ($\alpha = \pi/4$) in comparison with analytical curves. It can be concluded from Fig. 3 that the solution for the disk and Sirotin's solution for the rod are in good agreement with the numerical one, while Parfenov's solution deviates noticeably from it except for the neighbourhoods of $[001]$ and $[101]$ orientations, in which the two analytical solutions for the rod coincide.

It can also be seen from Fig. 3 that the simplified solution approximates well the complete Sirotin's solution, with the exception for the stresses σ_{jz} , which we neglected during simplification: The difference in solutions does not exceed units of percent and is generally less than the difference in Parfenov's and Sirotin's solutions.

Thus, we will consider below the analytical solution for the disk and Sirotin's solution for the rod sufficiently accurate for practical use. We obtained this result only for cubic crystals, but we assume that it is valid for crystals of any symmetry.

5. Calculation of thermo-optics in elastically anisotropic cubic crystals

The beam distortions induced by the active element with linear birefringence, independent of the longitudinal coordinate z , are determined by three functions of the transverse coordinates (x, y) : the arithmetic mean δ_0 of the phase incursions of the eigenpolarisations [subscripts I and II in expression (12)], the angle of incidence of these polarisations Ψ and difference in the incursions of their phases δ . Consider a beam that is

initially linearly polarised at an angle θ in the xy plane and passes through such an active element.

In the absence of circular birefringence, the ratio of the field intensity polarised after passing through one or more birefringent elements orthogonally to the initial beam polarisation to its total intensity is called the local degree of beam depolarisation. The integral degree of depolarisation is the ratio of the corresponding powers. In the plane wave approximation using the Jones matrix method [63], it is easy to obtain that after a beam passes through one active element, the resulting permittivity tensor of which (10) is independent of z , the local degree of thermally induced depolarisation can be calculated by the formula

$$\Gamma(x, y) = \sin^2(\delta/2) \sin^2 2(\Psi - \theta). \quad (53)$$

With weak birefringence, it simplifies to

$$\Gamma_0(x, y) \approx \frac{1}{4} \delta^2 \sin^2 2(\Psi - \theta) \quad (54)$$

and, as is easy to see, is determined by a linear combination of $\delta \cos 2\Psi$ and $\delta \sin 2\Psi$ with coefficients depending on the angle of inclination of the input polarisation.

The integral degree of depolarisation, by definition, is the local degree of depolarisation averaged over the input probe beam:

$$\gamma = \left[\iint_S |\mathbf{E}_{in}|^2 dS \right]^{-1} \iint_S \Gamma |\mathbf{E}_{in}|^2 dS, \quad (55)$$

where \mathbf{E}_{in} is the electric field strength (hereinafter, for simplicity, we assume that there is no absorption and amplification of probe radiation in the medium). With strong birefringence, γ tends to the steady-state value

$$\gamma_\infty = \left[\iint_S |\mathbf{E}_{in}|^2 dS \right]^{-1} \iint_S \gamma_{\infty loc} |\mathbf{E}_{in}|^2 dS, \quad (56)$$

where

$$\gamma_{\infty loc} = \langle \Gamma_\infty(x, y) \rangle = \frac{1}{2} \sin^2 2(\Psi - \theta) \quad (57)$$

(see, for example, [25]). The value of $\gamma_{\infty loc}$ is the local degree of depolarisation (53), averaged over a small region of the cross section, within the boundaries of which, under conditions of strong birefringence, the phase difference changes by a value much larger than 2π , while the temperature and stress fields, as well as the intensity profile of the probe beam, change weakly.

Using the same method, the phase distortion of the initially polarised beam component can be found in the form

$$\Delta\varphi(x, y) = \delta_0 + \arctan[\tan(\delta/2) \cos 2(\Psi - \theta)]. \quad (58)$$

In the approximation of the isotropic elasticity problem, it was shown that the dependence of thermally induced distortions of the beam on the orientation of the crystallographic axes in $m3m$ crystals is determined by the photoelastic anisotropy parameter ξ , which, depending on the choice between (3a) and (3b), can be written in the form of elasto-optic (ξ_p) or piezo-optic (ξ_π) relations [21, 22, 25]:

$$\xi_p = 2p_{66}/p_s = 2p_{abab}/p_s, \quad (59a)$$

$$\xi_\pi = \pi_{66}/\pi_S = 2\pi_{abab}/\pi_S \quad (59b)$$

[see the notation in (17)]. It is easy to show that [see (4)]

$$\xi_\pi = \xi_p \xi_s, \quad (60)$$

therefore, in elastically anisotropic crystals ($\xi_s \neq 1$), relations (59) differ. In m3 crystals, in addition to the parameter ξ , it is necessary to determine the second parameter of photoelastic anisotropy [25], which does not depend on the form of expression (3):

$$\xi_d = (\pi_{12} - \pi_{21})/\pi_S = (p_{12} - p_{21})/p_S. \quad (61)$$

Since we consider the description of the photoelastic effect in the form of (3b) to be preferable, and the tensor σ in elastically anisotropic crystals is completely or partially axially symmetric (see Section 4.2), the formulae for the thermally induced change in the permittivity will be similar to elastically isotropic expressions from [21, 22, 25], in which (59b) is used for ξ .

5.1. Beam distortion in elastically anisotropic cylindrical active elements

In accordance with formula (10b), the quantities Ψ , δ , and δ_0 can be written as

$$\begin{aligned} \cot 2\Psi(r, \varphi) &= \frac{\Delta B_{\sigma_{xx}} - \Delta B_{\sigma_{yy}}}{2\Delta B_{\sigma_{xy}}}, \\ \delta(r, \varphi) &= k_0 L(n_I - n_{II}) = -\frac{k_0 L n_0^3}{2} \frac{2\Delta B_{\sigma_{xy}}}{\sin 2\Psi} \\ &= -\frac{k_0 L n_0^3}{2} \frac{(\Delta B_{\sigma_{xx}} - \Delta B_{\sigma_{yy}})}{\cos 2\Psi}, \quad (62) \\ \delta_0(r, \varphi) &= k_0 L(n_0 - 1)u_{zz} + k_0 L[(n_I + n_{II})/2 - n_0] \\ &= k_0 L[(n_0 - 1)u_{zz} + \beta_\sigma(T - T_0) \\ &\quad - (n_0^3/2)(\Delta B_{\sigma_{xx}} + \Delta B_{\sigma_{yy}})/2]. \end{aligned}$$

Hereinafter, we assume for definiteness that the end faces of the element have no antireflective coatings, and the refractive index of the environment is equal to unity. Otherwise, the term in δ_0 , which is responsible for a change in the length of the element [proportional to $(n_0 - 1)u_{zz}$], should be corrected accordingly.

Substituting (3b), the stress tensor from Section 4.3.1, and the piezo-optic tensor of a cubic crystal (see Fig. 2) with an arbitrary orientation of the crystallographic axes into these expressions, in the geometry of the thin disk we obtain

$$\begin{aligned} \delta_0 &= \frac{p}{2Q_{\text{disk}}} \left\{ \beta_\sigma(1 - u) + \alpha_T(n_0 - 1) \right. \\ &\quad \times \left[(1 - u) + \frac{\nu_{[001]}}{Z_{\text{disk}}} \left(\frac{1}{2} - u \right) \right] + \alpha_T(n_0 - 1) \frac{1 + \nu_{[001]}}{Z_{\text{disk}}} \\ &\quad \times \left[F_{1s} \left(\frac{1}{2} - u \right) + \frac{u}{2} (F_{2s} \cos 2\varphi_\Phi + F_{3s} \sin 2\varphi_\Phi) \right] \left. \right\} \\ &+ \frac{p}{2Z_{\text{disk}}} \left[(1 + 2\xi_\pi + F_1) \left(\frac{1}{2} - u \right) + \frac{u}{2} (F_2 \cos 2\varphi_\Phi + F_3 \sin 2\varphi_\Phi) \right], \quad (63) \end{aligned}$$

$$\delta \cos 2\Psi = \frac{p}{Z_{\text{disk}}} \left[A_{1\Phi} \left(\frac{1}{2} - u \right) + \frac{u}{2} (A_{2\Phi} \cos 2\varphi_\Phi + A_{3\Phi} \sin 2\varphi_\Phi) \right],$$

$$\delta \sin 2\Psi = \frac{p}{Z_{\text{disk}}} \left[B_{1\Phi} \left(\frac{1}{2} - u \right) + \frac{u}{2} (B_{2\Phi} \cos 2\varphi_\Phi + B_{3\Phi} \sin 2\varphi_\Phi) \right],$$

where

$$p = QP_\Sigma / (\lambda\kappa) \quad (64)$$

is the dimensionless heat release power; the quantity u is defined in (21); λ is the wavelength in free space, and the expressions for $A_{i\Phi}$, $B_{i\Phi}$, F_i and F_{is} , $i = 1, \dots, 3$ are given in Appendix 3. Expressions (63) are given in the form most convenient for the case of weak birefringence.

With the same substitution using Sirotn's solution from Section 4.3.2, for a long rod we obtain

$$\begin{aligned} \delta_0 &= \frac{p}{2Q_{\text{rod}}} \left[\beta_\sigma(1 - u) + \frac{\alpha_T}{2}(n_0 - 1) \right] + \frac{p}{2Z_s} \left\{ (1 + 4\xi_\pi) \left(\frac{1}{2} - u \right) \right. \\ &\quad \left. - 2\xi_\pi \left[(D_1 - 1) \left(u - \frac{1}{2} \right) + \frac{u}{2} (D_2 \cos 2\varphi_\Phi + D_3 \sin 2\varphi_\Phi) \right] \right. \\ &\quad \left. + [(2D_1 - 1)F_1 - 2(D_b F_4 + D_c F_5)] \left(u - \frac{1}{2} \right) \right. \\ &\quad \left. + [F_2 + 2D_2 F_1 + 2(D_c F_5 - D_b F_4)] \frac{u}{2} \cos 2\varphi_\Phi \right. \\ &\quad \left. + [F_3 + 2D_3 F_1 + 2(D_c F_4 + D_b F_5)] \frac{u}{2} \sin 2\varphi_\Phi \right\}, \quad (65) \end{aligned}$$

$$\begin{aligned} \delta \cos 2\Psi &= \frac{p}{Z_s} \left\{ [(2D_1 - 1)A_{1\Phi} - 2(D_b A_{4\Phi} + D_c A_{5\Phi})] \left(u - \frac{1}{2} \right) \right. \\ &\quad \left. + [A_{2\Phi} + 2D_2 A_{1\Phi} + 2(D_c A_{5\Phi} - D_b A_{4\Phi})] \frac{u}{2} \cos 2\varphi_\Phi \right. \\ &\quad \left. + [A_{3\Phi} + 2D_3 A_{1\Phi} + 2(D_c A_{4\Phi} + D_b A_{5\Phi})] \frac{u}{2} \sin 2\varphi_\Phi \right\}, \end{aligned}$$

$$\begin{aligned} \delta \sin 2\Psi &= \frac{p}{Z_s} \left\{ [(2D_1 - 1)B_{1\Phi} - 2(D_b B_{4\Phi} + D_c B_{5\Phi})] \left(u - \frac{1}{2} \right) \right. \\ &\quad \left. + [B_{2\Phi} + 2D_2 B_{1\Phi} + 2(D_c B_{5\Phi} - D_b B_{4\Phi})] \frac{u}{2} \cos 2\varphi_\Phi \right. \\ &\quad \left. + [B_{3\Phi} + 2D_3 B_{1\Phi} + 2(D_c B_{4\Phi} + D_b B_{5\Phi})] \frac{u}{2} \sin 2\varphi_\Phi \right\}, \end{aligned}$$

where the expressions for $A_{i\Phi}$, $B_{i\Phi}$ and F_i ($i = 4, 5$) are also given in Appendix 3. To preserve the general form of the expressions for the elastically isotropic case, we introduced the thermo-optical constant Q as follows:

$$Q_{\text{disk}} = \frac{1}{4} \alpha_T n_0^3 \pi_S / s_{11} = \frac{1}{4} \alpha_T n_0^3 E_{[001]} \pi_S, \quad (66)$$

$$Q_{\text{rod}} = \frac{1}{4} \alpha_T n_0^3 \pi_S / (s_{11} + s_{22}) = Q_{\text{disk}} / (1 - \nu_{[001]}).$$

It is easy to see that in the elastically isotropic case (13), expressions (66) take the form of (16).

Expressions (63) and (65) are similar to the elastically isotropic ones; however, they have a number of important differences, in addition to defining Q . In a thin disk, the terms for the arithmetic mean of the phase incursions δ_0 , responsible for a change in the crystal length [proportional to $\alpha_T(n_0 - 1)$], now depend on the crystal orientation and, in the general

case, are not axially symmetric. In a rod, the corresponding term in our approximation is constant over the cross section of the active element. The terms in the phase incursions caused by the photoelastic effect have a factor of $1/Z$, which also depends on the crystal orientation (it is easy to show that Ψ is independent of Z). In a thin disk, this is the only difference in birefringence from the elastically isotropic case. Therefore, with weak birefringence, the degree of depolarisation will change in proportion to the square of the phase difference δ by Z_{disk}^{-2} times. With strong birefringence, when the degree of depolarisation γ_∞ (56) and (57) depends only on the angle of inclination of the eigenpolarisations Ψ , it will remain the same. In a long rod, the formulae for the elastically anisotropic case contain new terms that affect both δ and Ψ . Therefore, in this case, the dependence of the degree of depolarisation on the crystal orientation is a complex function of the parameters of photoelastic and elastic anisotropy.

In approximation (51), expressions (65) are simplified to the form:

$$\begin{aligned} \delta_0 \approx & \frac{p}{2Q_{\text{rod}}} \left[\beta_\sigma (1-u) + \frac{\alpha_T}{2} (n_0 - 1) \right] + \frac{p}{2Z_0} \left\{ (1 + 4\zeta_\pi) \left(\frac{1}{2} - u \right) \right. \\ & - 2\zeta_\pi \left[(D_{01} - 1) \left(u - \frac{1}{2} \right) - D_0 \frac{u}{2} (F_{2s} \cos 2\varphi_\Phi + F_{3s} \sin 2\varphi_\Phi) \right] \\ & + (2D_{01} - 1) F_1 \left(u - \frac{1}{2} \right) + \frac{u}{2} [(F_2 - 2D_0 F_{2s} F_1) \cos 2\varphi_\Phi \\ & \left. + (F_3 - 2D_0 F_{3s} F_1) \sin 2\varphi_\Phi] \right\}, \end{aligned} \quad (67)$$

$$\begin{aligned} \delta \cos 2\Psi & \approx \frac{p}{Z_0} \left[(2D_{01} - 1) A_{1\Phi} \left(u - \frac{1}{2} \right) + (A_{2\Phi} - 2D_0 F_{2s} A_{1\Phi}) \right. \\ & \left. \times \frac{u}{2} \cos 2\varphi_\Phi + (A_{3\Phi} - 2D_0 F_{3s} A_{1\Phi}) \frac{u}{2} \sin 2\varphi_\Phi \right], \\ \delta \sin 2\Psi & \approx \frac{p}{Z_0} \left[(2D_{01} - 1) B_{1\Phi} \left(u - \frac{1}{2} \right) + (B_{2\Phi} - 2D_0 F_{2s} B_{1\Phi}) \right. \\ & \left. \times \frac{u}{2} \cos 2\varphi_\Phi + (B_{3\Phi} - 2D_0 F_{3s} B_{1\Phi}) \frac{u}{2} \sin 2\varphi_\Phi \right]. \end{aligned}$$

Formulae (63), (65), and (67) can also be written differently: They will hold true if we omit the subscript Φ of the coefficients A_i and B_i and replace Ψ by

$$\Psi_\Phi = \Psi + \Phi. \quad (68)$$

In particular, at $\Phi = 0$, these formulae can be simplified by omitting the subscript Φ everywhere.

In the particular case of $\alpha = 0$ (orientations $[MON]$, see Fig. 1e), in a thin disk

$$\begin{aligned} \delta_0 = & \frac{p}{2Q_{\text{disk}}} \left\{ \beta_\sigma (1-u) + \alpha_T (n_0 - 1) \right. \\ & \times \left[(1-u) + \frac{v_{[001]}}{Z_{\text{disk}}} \left(\frac{1}{2} - u \right) \right] + \alpha_T (n_0 - 1) \frac{1 + v_{[001]}}{Z_{\text{disk}}} \\ & \times \left[F_{1s} \left(\frac{1}{2} - u \right) + F_{2s} \frac{u}{2} \cos 2\varphi_\Phi \right] \\ & \left. + \frac{p}{2Z_{\text{disk}}} \left[(1 + 2\zeta_\pi + F_1) \left(\frac{1}{2} - u \right) + F_2 \frac{u}{2} \cos 2\varphi_\Phi \right], \right\} \end{aligned} \quad (69)$$

$$\delta \cos 2\Psi_\Phi = \frac{p}{Z_{\text{disk}}} \left[A_1 \left(\frac{1}{2} - u \right) + A_2 \frac{u}{2} \cos 2\varphi_\Phi \right],$$

$$\delta \sin 2\Psi_\Phi = \frac{p}{Z_{\text{disk}}} \xi_\pi \frac{u}{2} \sin 2\varphi_\Phi,$$

and in a long rod, $D_b = D_3 = 0$, and

$$\begin{aligned} \delta_0 = & \frac{p}{2Q_{\text{rod}}} \left[\beta_\sigma (1-u) + \frac{\alpha_T}{2} (n_0 - 1) \right] + \frac{p}{2Z_s} \left\{ (1 + 4\zeta_\pi) \left(\frac{1}{2} - u \right) \right. \\ & - 2\zeta_\pi \left[(D_1 - 1) \left(u - \frac{1}{2} \right) + \frac{u}{2} D_2 \cos 2\varphi_\Phi \right] \\ & + [(2D_1 - 1) F_1 - 2D_c F_5] \left(u - \frac{1}{2} \right) \\ & \left. + (F_2 + 2D_2 F_1 + 2D_c F_5) \frac{u}{2} \cos 2\varphi_\Phi \right\}, \end{aligned} \quad (70)$$

$$\begin{aligned} \delta \cos 2\Psi_\Phi = & \frac{p}{Z_s} \left\{ [(2D_1 - 1) A_1 - 2D_c A_5] \left(u - \frac{1}{2} \right) \right. \\ & \left. + (A_2 + 2D_2 A_1 + 2D_c A_5) \frac{u}{2} \cos 2\varphi_\Phi \right\}, \end{aligned}$$

$$\delta \sin 2\Psi_\Phi = \frac{p}{Z_s} \xi_\pi \frac{u}{2} \sin 2\varphi_\Phi.$$

To avoid confusion, note that at $\beta = \pi/2$ (orientations $[NM0]$, equivalent to $[M0N]$) $D_c F_5 = D_c A_5 = 0$, but $D_b F_4$ and $D_b A_4$ are nonzero; otherwise, the formulae are the same. In approximation (51), expressions (70) take the form

$$\begin{aligned} \delta_0 \approx & \frac{p}{2Q_{\text{rod}}} \left[\beta_\sigma (1-u) + \frac{\alpha_T}{2} (n_0 - 1) \right] + \frac{p}{2Z_0} \left\{ (1 + 4\zeta_\pi) \left(\frac{1}{2} - u \right) \right. \\ & - 2\zeta_\pi \left[(D_{01} - 1) \left(u - \frac{1}{2} \right) + D_0 \frac{u}{2} F_{2s} \cos 2\varphi_\Phi \right] \\ & + (2D_{01} - 1) F_1 \left(u - \frac{1}{2} \right) + \frac{u}{2} (F_2 - 2D_0 F_{2s} F_1) \cos 2\varphi_\Phi \left. \right\}, \end{aligned} \quad (71)$$

$$\begin{aligned} \delta \cos 2\Psi_\Phi \approx & \frac{p}{Z_0} \left[(2D_{01} - 1) A_1 \left(u - \frac{1}{2} \right) \right. \\ & \left. + (A_2 - 2D_0 F_{2s} A_1) \frac{u}{2} \cos 2\varphi_\Phi \right], \end{aligned}$$

$$\delta \sin 2\Psi_\Phi \approx \frac{p}{Z_0} \xi_\pi \frac{u}{2} \sin 2\varphi_\Phi.$$

In $m3m$ crystals with $\alpha = \pi/4$ (orientations $[MMN]$, see Fig. 1f), in a thin disk

$$\delta \sin 2\Psi_\Phi = \frac{p}{Z_{\text{disk}}} B_3 \frac{u}{2} \sin 2\varphi_\Phi, \quad (72)$$

in a long rod, $D_b = D_3 = 0$ again, and

$$\delta \sin 2\Psi_\Phi = \frac{p}{Z_s} (B_3 + 2D_c B_4) \frac{u}{2} \sin 2\varphi_\Phi, \quad (73)$$

while in approximation (51)

$$\delta \sin 2\Psi_\Phi \approx \frac{P}{Z_0} B_3 \frac{u}{2} \sin 2\varphi_\Phi. \quad (74)$$

The values of δ_0 and $\delta \cos 2\Psi_\Phi$ in these cases are determined by expressions (69), (70), and (71) (the values of all coefficients should be recalculated taking into account the change in the angle α). We also recall that in the simplest orientations [001], [011], and [111], further simplification of expressions (69)–(74) is possible using formulae (A2.6).

5.2. Specific orientations in elastically anisotropic crystals

Mukhin et al. [22] argue that in an elastically isotropic m3m crystal in the absence of a critical [[C]] orientation, the simplest orientations related to symmetry elements of a crystal ([001], [011], and [111]) are the best and the worst from the point of view of minimising the degree of depolarisation (55). In the vast majority of cases, this statement is true. In all cases, a weaker statement holds: The simplest orientations are points of local extrema or saddle points of the degree of depolarisation. Due to the coincidence of the symmetries of the elastic and photoelastic tensors in m3m crystals (see Fig. 2a), allowance for the anisotropy of the elastic properties leaves these orientations extreme, although in rare cases the type of extremum can change. This case will be considered in the second part of the paper.

Unlike the simplest ones, the [[C]] orientation [18, 19] is not related to the symmetry elements of a crystal, and its position can change. The piezo-optic tensors of m3 crystals have a lower symmetry (Fig. 2b); therefore, orientations that are also not related to the symmetry elements of a crystal and are determined with good accuracy by the analytical estimates [[A]] and [[B]] [25] are optimal in most cases for these crystals. The position of these orientations can also change when anisotropy of the elastic properties of the crystal is taken into account. In Section 5.2, we calculated the position of [[A]], [[B]] and [[C]] orientations in elastically anisotropic crystals. In addition, we found the [[D]], which is also not related to the symmetry elements of the crystal and is a local minimum point or a saddle point, and in rare cases, a global minimum point of the degree of depolarisation under strong birefringence.

5.2.1. Orientation [[C]] in m3m crystals

Joiner et al. [18] calculated in the elastically isotropic approximation the [[C]] orientation position in m3m crystals for a long rod and a thin disk of arbitrary cross section with an arbitrary shape of the heating beam. The derivation is based on the requirement

$$\Delta B_{xy} = 0 \quad (75)$$

for $\Phi = 0$ and an arbitrary stress tensor within the framework of these geometries. Because $\alpha_{T,xy} \equiv 0$ in cubic crystals, $\Delta B_{\sigma,xy} = \Delta B_{u,xy}$ in (3). In this regard, the simultaneous use of (3b) for a thin disk (plane-stressed state) and (3a) for a long rod (plane-strained state) does not lead to errors. Otherwise, the derivation of the formulae in a disk and a rod is identical and leads to one of the [MMN] orientations (see Fig. 1f)

$$\alpha = \pi/4 + \pi k/2, \quad (76)$$

$$\tan^2 \beta = -\xi^{-1},$$

where $\xi = \xi_\pi$ in a disk and $\xi = \xi_p$ in a rod. It follows from this expression that with strong anisotropy of the elastic tensor, the critical orientations in a disk and in a rod of the same material should noticeably differ.

In an elastically anisotropic disk, solution [18] holds. However, as we noted in Section 4.3.4, in the case of a rod we need to use Sirotin's solution, which is not a plane strain (see Section 4.3.2), and, therefore, expression (76) is not applicable in this case. It is easy to obtain that in the set of [MMN] orientations ($\alpha = \pi/4$) considered in [18], for $\Phi = 0$ we have

$$\Delta B_{\sigma,xy} = \pi_{xyyz} \sigma_{yz} + \pi_{xyxy} \sigma_{xy},$$

$$\pi_{xyyz} = \frac{\pi_s \sin 2\beta (\xi_\pi - 1)}{4},$$

$$\pi_{xyxy} = \frac{\pi_s (\cos^2 \beta + \xi_\pi \sin^2 \beta)}{2}, \quad (77)$$

$$\pi_{xyxx} = \pi_{xyyy} = \pi_{xyzz} = \pi_{xyxz} = 0.$$

One can see that π_{xyxy} vanishes only under condition (76) with $\xi = \xi_\pi$, but in this case $\pi_{xyyz} \neq 0$. Since in Sirotin's solution $D_c \neq 0$ for these orientations (see Fig. 3b) and, therefore, $\sigma_{yz} \neq 0$, in the elastically anisotropic case the existence and position of the critical orientation depend on the stress tensor. Substituting (24b) and (31) into (77) and requiring the fulfilment of condition (75), we obtain the transcendental equation for the angle β :

$$\cos 2\beta - 2D_c(\beta) \sin 2\beta = \frac{\xi_\pi + 1}{\xi_\pi - 1}. \quad (78)$$

The critical orientation in an elastically anisotropic rod, defined by this equation, will be denoted by [[C_s]]. Because, as follows from our comparison of relations (49), the coefficient D_c is small compared to a similar value obtained from the strain tensor, we expect that [[C_s]] is slightly different from the critical disk orientation, which coincides with [[C]], and is much further from position (76) with $\xi = \xi_p$ predicted in [18] (see the second part of the paper). Since in the simplified Sirotin's solution we completely neglected this component of the stress tensor, the resulting critical orientation coincides with that for a disk. Due to the fact that in the specific orientations [001], [111], and [011], $D_c = 0$ (see A2.6), the orientations [[C_s]] and [[C]] coincide with them under the same values of ξ_π ($-\infty$, -0.5 and 0 , respectively). Because in the elastically isotropic case two of them ([001] and [011]) are marginal for the existence of the critical orientation, the domain of its determination does not change when the anisotropy of the elastic properties is taken into account.

5.2.2. Orientation [[C]] in m3 crystals

Critical orientation can also exist in m3 crystals, as in m3m crystals. In our work [25], we showed that in the elastically isotropic approximation, the [[C]] orientation exists for

$$\begin{aligned} \xi < 0, \\ (-\xi - 3)\xi_d^2 < 4(1 - \xi), \end{aligned} \quad (79)$$

and its position is determined by a cumbersome system of transcendental equations. For $\xi_d \neq 0$ in the notations of

$$\begin{aligned} X &= \cos 2\alpha \sin^2 \beta, \\ Y &= \cos^2 \beta \end{aligned} \quad (80)$$

we managed to reduce these equations to a system

$$\begin{aligned} 3(\kappa_d Y - q)[q + (2Y - 1)\kappa_d]^2 &= q^2(\kappa_d - 1)[3q + (9Y - 4)\kappa_d], \\ 2X[q + (2Y - 1)\kappa_d] &= q_d q(1 - 3Y), \end{aligned} \quad (81)$$

where

$$q = \xi_\pi / (\xi_\pi - 1); \quad q_d = \xi_d / (\xi_\pi - 1); \quad \kappa_d = 1 + 3q_d^2/4. \quad (82)$$

System (81) has three solutions corresponding to physically equivalent orientations of the general form $[MNP]$, $[NPM]$ and $[PMN]$. A numerical analysis showed that outside the domain (79), the solutions to system (81) are complex, that is, the $[[C]]$ orientation does not exist. Note that in the degenerate case of the m3m crystal ($\xi_d = 0$), system (81) allows one to find only solution (76), and for the other two roots the values of X and, accordingly, the angle α are not determined (these solutions are given in [25]).

As in m3m crystals, the critical orientation in an elastically anisotropic disk coincides with $[[C]]$. The critical orientation in an elastically anisotropic rod also coincides with it in a simplified solution and slightly deviates ($[[C_s]]$) in the exact one (see the second part of the paper). In the latter case, the existence of a critical orientation in the strict sense has not been proved; however, for some values of the Euler angles, numerical calculations demonstrate a significant decrease in the degree of depolarisation, which is characteristic of the critical orientation.

5.2.3. Orientation $[[A]]$ in m3 crystals

In our work [25], we determined two more orientations ($[[A]]$ and $[[B]]$) of the form $[MON]$ (see Fig. 1e) in elastically isotropic m3 crystals. In the $[001]$ orientation of the m3m crystals, Ψ does not depend on r , since in expressions (63), (65), and (67) $A_{1\phi} = B_{1\phi} = 0$ (see Ref. [22] and (A3.2)–(A3.4)). In m3 crystals, this condition is not met because of the nonzero coefficient A_1 , which vanishes in the $[[A]]$ orientation, determined, for example, by the Euler angles

$$\alpha = 0, \quad \cos 2\beta = q_d(1 - \sqrt{1 + q_d^{-2}}). \quad (83)$$

In the particular case of $\xi_d = 0$, the $[[A]]$ orientation coincides with $[001]$, and for $|\xi_d| \gg |\xi_\pi - 1|$, we observe $[[A]] \rightarrow [101]$. In elastically anisotropic crystals, this calculation is valid for a disk and in approximation (51), for a rod. If approximation (51) does not hold in a rod, instead of $A_1 = 0$, it is necessary to require that

$$(2D_1 - 1)A_1 - 2(D_b A_4 + D_c A_5) = 0 \quad (84)$$

(in orientations of the form $[MON]$, $[OMN]$, etc. A_4 and D_b or A_5 and D_c always vanish). Condition (84) is satisfied in some $[[A_s]]$ orientation also of the form $[MON]$, slightly different from (83).

In [25], we found that for weak birefringence, and also for a large probe beam with strong birefringence, the degree of depolarisation in m3 crystals is usually minimal in an orientation close to $[[A]]$ (see the second part of the paper). For simplicity, unless otherwise specified, we will neglect this distinction.

5.2.4. Orientation $[[B]]$ in m3 crystals

The $[011]$ orientation of m3m crystals is interesting in that the angle of inclination of the eigenpolarisations Ψ in the central region of the active element turns out to be weakly dependent on the coordinates, which allows one to obtain relatively small depolarisation in the regime of strong birefringence for beams of small radii (see Ref. [24] and second part of the paper). The degree of depolarisation in this regime is approximated by expressions (56) and (57), which depend on Ψ and do not depend on δ .

The angle of inclination of eigenpolarisations Ψ in cylindrical active elements in various approximations (see [22, 25], (63), (65), and (67)) has the general form

$$\cot 2\Psi = \tilde{A}/\tilde{B}, \quad (85a)$$

where

$$\begin{aligned} \tilde{A} &= \tilde{A}_1 g(r) + (\tilde{A}_2 \cos 2\varphi_\phi + \tilde{A}_3 \sin 2\varphi_\phi) h(r); \\ \tilde{B} &= \tilde{B}_1 g(r) + (\tilde{B}_2 \cos 2\varphi_\phi + \tilde{B}_3 \sin 2\varphi_\phi) h(r); \end{aligned} \quad (85b)$$

and at $r \rightarrow 0$, we have $h(r) \rightarrow 0$ and $g(r) \rightarrow \text{const}$. To minimise the degree of depolarisation, we choose the input polarisation so that $\theta = \Psi(r = 0)$, then (57), taking into account (85), will have the form:

$$\gamma_{\infty \text{ loc}} = \frac{(\tilde{A}\tilde{B}_1 - \tilde{B}\tilde{A}_1)^2}{(\tilde{A}^2 + \tilde{B}^2)(\tilde{A}_1^2 + \tilde{B}_1^2)}. \quad (86)$$

Consider the case of a small diameter of the probe beam, within which $h \ll g$. In the first approximation, the integral degree of depolarisation (56) will be as follows:

$$\gamma_{\infty} \approx \frac{I_{\infty}}{2 \int_0^R |\mathbf{E}_{\text{in}}|^2 r dr} \int_0^R \frac{h^2}{g^2} |\mathbf{E}_{\text{in}}|^2 r dr, \quad (87)$$

$$I_{\infty} = \frac{(\tilde{A}_2 \tilde{B}_1 - \tilde{B}_2 \tilde{A}_1)^2 + (\tilde{A}_3 \tilde{B}_1 - \tilde{B}_3 \tilde{A}_1)^2}{(\tilde{A}_1^2 + \tilde{B}_1^2)^2}.$$

Since in this calculation, the change in the angle Φ is compensated for by the automatic selection of the angle θ , the value of I_{∞} depends only on the orientation of the crystal and its elastic and photoelastic properties, and we can assume without loss of generality that $\Phi = 0$. In the critical orientation, I_{∞} , by definition, vanishes. A numerical analysis shows that apart from this I_{∞} and, together with it, γ_{∞} have a minimum or a saddle point in one of the orientations of the form $[MON]$ (see Fig. 1e). In these orientations at $\Phi = 0$, the coefficient $\tilde{B}_1 = \tilde{B}_2 = 0$; therefore,

$$I_{\infty [M0N]} = (\tilde{B}_3^2 / \tilde{A}_1^2) \Big|_{\phi=0}. \quad (88)$$

In particular, in the elastically isotropic approximation and in the geometry of a thin disk

$$I_{\infty [M0N]0} = \xi_{\pi}^2 / A_1^2. \quad (89)$$

This quantity has a minimum in the sought-for orientation, called $[[B]]$, determined, for example, by the Euler angles [25]

$$\alpha = 0, \quad (90)$$

$$\cos 2\beta = q_d.$$

In a particular case of $\xi_d = 0$ (in a $m3m$ crystal), the $[[B]]$ orientation coincides with $[101]$, and if $|\xi_d| = |\xi_{\pi} - 1|$, then $[[B]]$ is the same as $[001]$. For convenience, we will ascertain, in contrast to [25], that for $|\xi_d| > |\xi_{\pi} - 1|$ this orientation does not exist. In this case, the minimum of $I_{\infty [M0N]0}$ is reached in the $[001]$ orientation.

In [25], we found that the degree of depolarisation for a thin probe beam with strong birefringence in $m3$ crystals is usually minimal in an orientation close to $[[B]]$ (see the second part of the paper). For simplicity, unless otherwise specified, we will neglect this distinction.

For a negative ξ_{π} , small in absolute value, the $[[B]]$ orientation, being close to the critical orientation, is not a minimum point of the dependence of the degree of depolarisation on the Euler angles (α, β) , but a saddle point. In particular, in elastically isotropic $m3m$ crystals this occurs at $-1/3 < \xi_{\pi} < 0$.

In the case of a long rod, expression (88) in compliance with (51) takes the form

$$I_{\infty [M0N]s0} = \xi_{\pi}^2 [(2D_{01} - 1)A_1]^{-2}. \quad (91)$$

Substituting (A3.3) into (91) allows one to convert it into

$$I_{\infty [M0N]s0} = G_1^2 / J_{[M0N]s0}^2, \quad W = \cos 2\beta, \quad (92)$$

$$J_{[M0N]s0} = \frac{2q_d W + 1 - W^2}{2 - \eta(1 - W^2)}, \quad \eta = (1 + \nu_{[001]})(1 - \xi_s),$$

where the constant G_1 is independent of orientation, and due to (38)

$$-\infty < \eta \leq 3/2. \quad (93)$$

As a result of studying the function $J_{[M0N]s0}(W)$ for an extremum, we obtain a quadratic equation. An analysis of its roots shows that $I_{\infty [M0N]s0}$ is minimal in the orientation, which we will call $[[B_{\sim}]]$, determined, for example, by the Euler angles

$$\alpha = 0, \quad (94)$$

$$\cos 2\beta = (\eta q_d)^{-1} \{ [1 - q_d^2 \eta (\eta - 2)]^{1/2} - 1 \},$$

which, taking into account (93), exists when

$$(1 - \eta)|q_d| \leq 1, \quad (95)$$

and coincides with $[001]$ when (95) turns into equality. Outside the domain of $[[B_{\sim}]]$ existence, the value of $I_{\infty [M0N]s0}$ is mini-

mal in the $[001]$ orientation. For $\eta = 0$ ($\xi_s = 1$) $[[B_{\sim}]]$ coincides with $[[B]]$, and in $m3m$ crystals (for $q_d = 0$) $[[B_{\sim}]]$ coincides with $[101]$.

In the case of strongly anisotropic elastic properties, condition (51) is not met, and among a number of orientations ($[M0N]$, $[0MN]$, etc.), it is necessary to minimise the quantity

$$I_{\infty [M0N]s} = \xi_{\pi}^2 [(2D_1 - 1)A_1 - 2(D_b A_4 + D_c A_5)]^{-2}. \quad (96)$$

It has a minimum in some $[[B_s]]$ orientation, slightly different from $[[B_{\sim}]]$, if one exists. Otherwise, it has a minimum in the $[001]$ orientation.

5.2.5. Orientation $[[D]]$ in $m3m$ and $m3$ crystals

In $m3m$ crystals, the value of I_{∞} (87) turns into $+\infty$ and, therefore, has points of maximum in $[001]$ and $[111]$ orientations. Due to the fact that the integral degree of depolarisation in the crystals of $[M'M'N]$ and $[M''M'N]$ orientations is the same (see [25]), the value of I_{∞} must have another point of minimum or a saddle point in one of orientations of the form $[MMN]$, where $M < N$. In these orientations, it is similar to (88)

$$I_{\infty [MMN]m3m} = (\tilde{B}_3^2 / \tilde{A}_1^2) \Big|_{\phi=0}. \quad (97)$$

In particular, in the elastically isotropic approximation and in the geometry of a thin disk

$$I_{\infty [MMN]m3m0} = 16/J_{[MMN]0}^2, \quad (98a)$$

where

$$J_{[MMN]0} = \frac{(1 - Y)(1 - 3Y)}{Y - q}, \quad (98b)$$

and the quantity Y is defined in (80). The denominator of $J_{[MMN]0}$ vanishes in the $[[C]]$ orientation. Investigation of the function $J_{[MMN]0}(Y)$ for other extrema, as in the case of $J_{[M0N]s0}(W)$, reduces to a quadratic equation. An analysis of its roots shows that $I_{\infty [MMN]m3m0}$ is minimal in the orientation, which we will call the $[[D]]$ orientation, determined by the Euler angles

$$\alpha = \pi/4 + \pi k/2, \quad (99)$$

$$\cos^2 \beta = \frac{\xi_{\pi} - [(2\xi_{\pi} + 1)/3]^{1/2}}{\xi_{\pi} - 1}.$$

This orientation exists for $\xi_{\pi} > -1/2$ and is a point of minimum of $I_{\infty}(\alpha, \beta)$ for $\xi_{\pi} > \xi_0[[D]]$, where $\xi_0[[D]] \approx 1.65$, and for lower values, it is a saddle point. The value of I_{∞} in the $[[D]]$ orientation becomes smaller than in $[011]$ for $\xi_{\pi} > \xi[[D]]$, where $\xi[[D]] \approx 10$. In this domain, the $[[D]]$ orientation may be better than $[011]$ in the case of strong birefringence. In a disk at $\xi_{\pi} < 0$, in order to minimise the degree of depolarisation, it is better to use the $[[C]]$ orientation; in a rod, it is necessary to take into account the important limitation considered at the end of the second part of the paper.

In a long rod with anisotropic elasticity, expression (97) in compliance with (51) takes the form

$$I_{\infty [MMN]m3ms0} = B_3^2 [(2D_{01} - 1)A_1]^{-2}. \quad (100)$$

Substituting (A3.3) into (100) allows one to convert it into

$$I_{\infty [MMN] m3m s0} = G_2^2 / J_{[MMN] s0}^2, \quad (101a)$$

where

$$J_{[MMN] s0} = [1 - (1 - Y)(1 + 3Y)\eta/2] J_{[MMN] 0}; \quad (101b)$$

and the constant G_2 is independent of orientation. The extremum condition for $J_{[MMN] s0}(Y)$ reduces to an equation of the 4th degree of general form. The orientation in which this quantity has a minimum is called the $[[D_{\sim}]]$ orientation. Since in approximation (51) $\eta \ll 1$, we expect that this orientation will be close to $[[D]]$. Finally, in the case of strongly anisotropic elastic properties, it is necessary to minimise the quantity

$$I_{\infty [MMN] s} = \left[\frac{B_3 + 2D_c B_4}{(2D_1 - 1)A_1 - 2D_c A_5} \right]^2 \quad (102)$$

among the $[MMN]$ orientations. The numerical calculation showed that it has a minimum in some $[[D_s]]$ orientation, displaced relative to $[[D]]$ and $[[D_{\sim}]]$.

A numerical analysis of the degree of depolarisation γ_{∞} in the elastically isotropic approximation showed that the orientation $[[D]]$ is the best in two cases. First, for $\xi > 10$ and a very small radius of the probe beam (approximately $0.2R$ or less for a flat-top profile), the gain in the degree of depolarisation compared with $[011]$ can be significant; as the radius of the probe beam increases, the $[011]$ orientation becomes preferable. Second, there is a narrow range, $1.2 < \xi < 1.7$, in which the orientation close to $[[D]]$, but not coinciding with it, is slightly better than $[011]$ and $[001]$ orientations for beams with a radius of about $0.5R$.

Thus, the conditions in which the use of the $[[D]]$ orientation is advantageous are very specific; therefore, the practical use of crystals of this orientation is unlikely. In this connection, the $[[D]]$ orientation will not be used in the calculations in the second part of the paper.

In $m3$ crystals, the search for the $[[D]]$ orientation is significantly complicated by the fact that, like $[[C]]$, it is an orientation of the general form $[MNP]$, and, due to the low practical value of this orientation, the search was not analytically carried out. A numerical analysis of the value of I_{∞} shows that the $[[D]]$ orientation, like $[[C]]$, disappears when the parameter $|\xi_d|$ exceeds some critical level, merging either with $[[B]]$, or with $[001]$.

Thus, in the geometry of a thin disk, $[[A]]$, $[[B]]$, $[[C]]$, and $[[D]]$ orientations, not related to the symmetry elements of the crystal, coincide with orientations in an elastically isotropic material. In the geometry of a long rod, in the approximation of weak anisotropy, $[[B]]$ and $[[D]]$ orientations differ, and in the case of strong anisotropy, all four orientations differ. For convenience, the numbers of formulae characterising these specific orientations are given in Table 2.

Table 2. Specific orientations of cubic crystals, not related to the elements of their symmetry.

Symmetry	Isotropic elasticity or a thin disk	Long rod with weak anisotropy	Long rod with arbitrary anisotropy
m3	$[[A]]$ (83)	$[[A]]$ (83)	$[[A_s]]$ (84)
	$[[B]]$ (90)	$[[B_{\sim}]]$ (94), (95)	$[[B_s]]$ (96)
	$[[C]]$ (79), (81)	$[[C]]$ (79), (81)	$[[C_s]]$ –
	$[[D]]$ –	$[[D_{\sim}]]$ –	$[[D_s]]$ –
m3m	$[[C]]$ (76), $\xi = \xi_{\pi}$	$[[C]]$ (76), $\xi = \xi_{\pi}$	$[[C_s]]$ (78)
	$[[D]]$ (99)	$[[D_{\sim}]]$ (101)	$[[D_s]]$ (102)

5.3. Determination of the effective values of thermo-optical constants Q and P in various crystal orientations

In measurements of thermally induced birefringence in optical media, Q is often used not only and not so much in the sense of definition of (16) but as a material constant characterising the degree of depolarisation in single crystals of individual orientations and in ceramics [48, 64]. Consider this application in more detail.

5.3.1. Thermo-optical constant Q

It follows from (53) that local (Γ) and integral (γ) degrees of depolarisation are periodic functions of the angle of inclination of the probe beam polarisation θ with a period $\pi/2$. Consequently, γ is periodic in the angle of the crystal rotation Φ relative to the z axis and has the same period. The integral degree of depolarisation for weak birefringence in the particular case of $\theta = 0$ in an elastically isotropic $m3$ crystal of $[001]$ orientation ($\alpha = 0, \beta = 0$) has extrema at $\Phi = 0$ and $\pi/4$, and the approximation of the local degree of depolarisation, quadratic in the heat release power,

$$\Gamma_0 = \frac{1}{4}(\delta \sin 2\Psi)^2 \quad (103)$$

is such that

$$\Gamma_{0 m3m}([001], \Phi = \pi/4) = Q^2 / (4\lambda\kappa)^2 P_{\Sigma}^2 u^2 \sin^2 2\varphi, \quad (104)$$

$$\Gamma_{0 m3m}([001], \Phi = 0) = \xi^2 \Gamma_{0 m3m}([001], \Phi = \pi/4).$$

It depends on the medium parameters Q/κ and ξ (recall that Q is different in a disk and in a rod). In order to reduce the degree of depolarisation with weak birefringence in the $[111]$ orientation to the same form, which for any Φ is

$$\Gamma_{0 m3m}([111]) = \left(\frac{Q}{4\lambda\kappa} \frac{1 + 2\xi}{3} \right)^2 P_{\Sigma}^2 u^2 \sin^2 2\varphi, \quad (105)$$

we introduce the quantities [48]

$$Q_{[111] m3m iso}^{eff} = \frac{(1 + 2\xi)Q}{3}, \quad (106)$$

$$\xi_{[111] m3m iso}^{eff} = 1.$$

In other orientations of $m3m$ crystals, it is generally impossible to introduce these quantities, since the degree of depolarisation cannot be reduced to (104) for two values of Φ differing by 45° (see Theorem 4 in [22]).

In the elastically anisotropic case, in order to reduce the degree of depolarisation to (104) in the $[001]$ orientation of a $m3m$ crystal, it is necessary to introduce

$$Q_{[001]m3m}^{\text{eff}} = Q/Z([001]),$$

$$\xi_{[001]m3m}^{\text{eff}} = \xi_{\pi},$$
(107)

where Q and Z are different in a disk and a rod. The formula for a rod was obtained in [65]. In m3 crystals, the orientation $[[A]]$, in which

$$Q_{[[A]]}^{\text{eff}} = A_{2[[A]]} Q/Z([[A]]),$$

$$\xi_{[[A]]}^{\text{eff}} = \xi_{\pi}/A_{2[[A]]},$$
(108)

$$A_{2[[A]]} = 1 - \frac{1}{2}(1 - \xi_{\pi})(\sqrt{q_d^4 + q_d^2} - q_d^2)$$

is an analogue of the [001] orientation. In the particular case of $\xi_d = 0$, the $[[A]]$ orientation coincides with [001], and $A_{2[[A]]} = 1$. If $|\xi_d| \gg |\xi_{\pi} - 1|$, then $[[A]] \rightarrow [101]$, and $A_{2[[A]]} \rightarrow (\xi_{\pi} + 3)/4$. If approximation (51) is not observed in the rod, the values of Q^{eff} and ξ^{eff} corresponding to the $[[A_s]]$ orientation will differ from those calculated by formula (108).

In the [111] orientation, with weak birefringence and any Φ , the degree of depolarisation has the form

$$F_0([111]) = \left(\frac{Q}{4\lambda\kappa}\right)^2 \frac{P_{\Sigma}^2}{Z([111])^2}$$

$$\times \left[\left(\frac{1 + 2\xi_{\pi}}{3}\right)^2 + \frac{\xi_d^2}{12} \right] u^2 \sin^2 2(\varphi \pm \chi_d),$$
(109)

where

$$\tan 2\chi_d = \sqrt{3} \xi_d / (2 + 4\xi_{\pi}),$$
(110)

and the choice of sign depends on which of the eight equivalent orientations ([111], $[1\bar{1}\bar{1}]$, $[1\bar{1}1]$, etc.) is selected. This expression cannot be reduced to the form of (104), but it is most similar to it when

$$Q_{[111]}^{\text{eff}} = \left[\left(\frac{1 + 2\xi_{\pi}}{3}\right)^2 + \frac{\xi_d^2}{12} \right]^{1/2} \frac{Q}{Z([111])},$$

$$\xi_{[111]}^{\text{eff}} = 1,$$
(111)

differing only in the nonzero value of χ_d .

Note that there is no physical difference between two opposite ξ_d values, since when a cubic crystal with parameters $(\xi_{\pi}, \xi_d, \xi_s)$ rotates by $\pm 90^\circ$ relative to any of [001], [010] and [100] directions, after a change of notations of the crystallographic axes we obtain a crystal with $(\xi_{\pi}, -\xi_d, \xi_s)$.

For convenience, all special cases in which Q^{eff} and ξ^{eff} can be introduced are listed in Table 3. Note that it is possible to

Table 3. Formulae for the parameters Q^{eff} and ξ^{eff} .

Orientation	$\xi_s = 1, m3m$	$\xi_s = 1, m3$	m3m	m3
[001]	Q, ξ	–	(107)	–
[[A]]	$\equiv [001]$	(108) при $Z = 1$	$\equiv [001]$	(108)
[111]	(106)	(111) при $Z = 1$	(111) при $\xi_d = 0$	(111)

Note. In a long rod, formulae for the $[[A]]$ orientation are valid in approximation (51).

introduce these quantities in an arbitrary orientation, for example, in a counter-rotation scheme [66], but they will not be reduced to those considered here (in particular, they will not depend on ξ_d) and will not make sense for a single active element.

5.3.2. Thermo-optical constant P

Due to the factor $1/Z$ present in the photoelastic terms of the phase incursions in (63), (65), and (67), but absent in the terms responsible for $\partial n/\partial T$, the thermo-optical constant P in the case of anisotropy of elastic properties cannot be introduced in a form analogous to (15) as a material parameter that allows one to simplify the expressions for δ_0 , as was done, for example, in [30, 32, 33, 35, 48]. The orientation-dependent P^{eff} can be determined by formula (12), but, as in elastically isotropic m3m crystals [64], only for two orientations in which δ_0 is axially symmetric, i.e. in this case, for [111] and $[[A']]$. The latter differs from $[[A]]$ (83) by the sign of q_d and similarly to it in the case of an m3m crystal is identical to [001], and in the case of a rod made of an m3 crystal it is analytically determined only in approximation (51). To determine P^{eff} for an arbitrary orientation, we generalise expression (12) by averaging the refractive indices over the polar angle:

$$\int_0^{2\pi} [n(r) + n_{II}(r)] d\varphi / (4\pi) = n(0) + [T(r) - T(0)] P^{\text{eff}}. \quad (112)$$

The quantity that is obtained upon inclusion of the term responsible for a change in the active element length (see [11, 16]) into P^{eff} is denoted by P^{tot} . It is defined by the expression

$$\int_0^{2\pi} \delta_0(r) d\varphi / (2\pi) = \delta_0(0) + k_0 L [T(r) - T(0)] P^{\text{tot}}. \quad (113)$$

In a long rod, neglecting the end effects, P^{eff} and P^{tot} are identical.

Using (63), (65) and (67), for arbitrary orientation we obtain

$$P_{\text{disk}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{disk}}}{Z_{\text{disk}}} (1 + 2\xi_{\pi} + F_1),$$

$$P_{\text{disk}}^{\text{tot}} = P_{\text{disk}}^{\text{eff}} + \alpha_T (n_0 - 1) \left(1 + \frac{\nu_{[001]}}{Z_{\text{disk}}} + \frac{1 + \nu_{[001]}}{Z_{\text{disk}}} F_{1s} \right),$$

$$P_{\text{rod}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{rod}}}{Z_s} [1 + 4\xi_{\pi} + 2\xi_{\pi}(D_1 - 1)] \quad (114)$$

$$- (2D_1 - 1)F_1 + 2(D_b F_4 + D_c F_5)]$$

$$\approx \beta_{\sigma} + \frac{Q_{\text{rod}}}{Z_0} [1 + 4\xi_{\pi} + 2\xi_{\pi}(D_{01} - 1) - (2D_{01} - 1)F_1].$$

Note that only coefficients F_4 and F_5 in this expression depend on ξ_d . In the specific orientations, expression (114) can be simplified. In particular, in the orientation [001]

$$P_{\text{disk}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{disk}}}{Z_{\text{disk}}}(1 + 2\zeta_{\pi}),$$

$$P_{\text{disk}}^{\text{tot}} = P_{\text{disk}}^{\text{eff}} + \alpha_T(n_0 - 1)\left(1 + \frac{\nu_{[001]}}{Z_{\text{disk}}}\right), \quad (115a)$$

$$P_{\text{rod}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{rod}}}{Z_s}\left(1 + 4\zeta_{\pi} + \zeta_{\pi} \frac{\xi_s - 1}{2}\right);$$

in the [011] orientation

$$P_{\text{disk}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{disk}}}{Z_{\text{disk}}}\left(1 + 2\zeta_{\pi} + \frac{\xi_{\pi} - 1}{4}\right),$$

$$P_{\text{disk}}^{\text{tot}} = P_{\text{disk}}^{\text{eff}} + \alpha_T(n_0 - 1)\left(1 + \frac{\nu_{[001]}}{Z_{\text{disk}}} + \frac{1 + \nu_{[001]}}{Z_{\text{disk}}} \frac{\xi_s - 1}{4}\right), \quad (115b)$$

$$P_{\text{rod}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{rod}}}{Z_s}\left[1 + 4\zeta_{\pi} + 2\zeta_{\pi}(D_{[011]} - 1) - (2D_{[011]} - 1) \frac{\xi_{\pi} - 1}{4}\right];$$

and in the [111] orientation

$$P_{\text{disk}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{disk}}}{Z_{\text{disk}}}\left(1 + 2\zeta_{\pi} + \frac{\xi_{\pi} - 1}{3}\right),$$

$$P_{\text{disk}}^{\text{tot}} = P_{\text{disk}}^{\text{eff}} + \alpha_T(n_0 - 1)\left(1 + \frac{\nu_{[001]}}{Z_{\text{disk}}} + \frac{1 + \nu_{[001]}}{Z_{\text{disk}}} \frac{\xi_s - 1}{3}\right), \quad (115c)$$

$$P_{\text{rod}}^{\text{eff}} = \beta_{\sigma} + \frac{Q_{\text{rod}}}{Z_s}\left[1 + 4\zeta_{\pi} + \zeta_{\pi} \frac{\xi_s^{-1} - 1}{3} - (2 + \xi_s^{-1}) \frac{\xi_{\pi} - 1}{9}\right].$$

In these expressions, for calculating Z_{disk} one needs to use (39), and for calculating Z_s (A2.6). In [[A]], [[A']], [[B]], and [[B~]] orientations, as well as in m3m crystals of [[C]] and [[D]] orientations, the expressions for P^{eff} and P^{tot} are obtained [approximation (51) is implied in a rod] by substituting the quantities

$$f_{1[[A]]} = f_{1[[A']]} = \frac{1}{2}q_{\text{d}}^2(1 - \sqrt{1 + q_{\text{d}}^2}),$$

$$f_{1[[B]]} = -\frac{1}{4}(1 - q_{\text{d}}^2),$$

$$f_{1[[B~]]} = [1 - q_{\text{d}}^2\eta(\eta - 1) - \sqrt{1 - q_{\text{d}}^2\eta(\eta - 2)}]/(2q_{\text{d}}^2\eta^2), \quad (116)$$

$$f_{1[[C]]m3m} = \left(\xi_{\pi} - \frac{1}{4}\right)/(1 - \xi_{\pi})^2,$$

$$f_{1[[D]]m3m} = \frac{3\xi_{\pi} - (2\xi_{\pi} + 1)^{3/2}/\sqrt{3}}{2(1 - \xi_{\pi})^2}$$

into (114), respectively. These quantities are used as a factor f_1 in the functions F_1 , F_{1s} , Z_{disk} and D_{01} .

In elastically isotropic media, (114) simplifies to well-known expressions [30, 32, 33, 35, 48]

$$P_{\text{disk}}^{\text{eff}} = P_{\text{disk}} + Q_{\text{disk}}F_1,$$

$$P_{\text{disk}}^{\text{tot}} = P_{\text{disk}}^{\text{eff}} + \alpha_T(n_0 - 1)(1 + \nu_{[001]}), \quad (117)$$

$$P_{\text{rod}}^{\text{eff}} = P_{\text{rod}} - Q_{\text{rod}}F_1,$$

where P_{disk} and P_{rod} are defined in (15).

6. Conclusions

We have studied thermally induced distortions of a beam in cylindrical active elements in the form of a long rod and a thin disk made of single cubic syngony crystals of all symmetry groups with an anisotropic elastic stiffness tensor under uniform pumping and lateral heat removal. It is shown that the description of the photoelastic effect using the piezo-optic tensor is preferable to the description using the elasto-optic tensor. Existing analytical solutions to the problems of heat conduction and elasticity in anisotropic media of arbitrary symmetry are reviewed. In a long rod made of a cubic crystal, a choice is made between two solutions to the elasticity problem in favour of a solution that does not correspond to plane strain; for weak anisotropy of the elastic stiffness tensor, a simplified solution of the elasticity problem is found.

For an arbitrary orientation of the crystallographic axes, the arithmetic mean and the difference between thermally induced phase incursions of eigenpolarisations, as well as the angle of inclination of these polarisations, are calculated. The specific orientation [[D]] is introduced. It is shown that the specific orientations [[A]], [[B]], [[C]], and [[D]], not related to crystal symmetry elements, are the same both in a disk of an elastically anisotropic crystal and in the elastically isotropic approximation. In a rod, in the approximation of weak anisotropy of the elastic stiffness tensor, [[B]] and [[D]] orientations differ from them, and in the case of strong anisotropy, all four orientations differ. Effective values of the thermo-optical constant Q are found in two specific orientations ([[A]] and [111]). The definition of the effective value of the thermo-optical constant P is generalised; its value is found in an arbitrary orientation.

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Appendix 1

This appendix provides an overview of the methods for writing the photoelastic additive to the permittivity tensor used in the literature. Thermo-optics is not considered in book [5], and Hooke's law in the chapters on the elastic properties of crystals and the photoelastic effect is written without the temperature term:

$$u_{ij} = s_{ijkl}\sigma_{kl} \quad (A1.1)$$

[formula (6) is present in the chapter on the thermodynamics of equilibrium states in crystals]. In paper [7] devoted to long glass rods, both forms of (3) are used, and the transition between them is carried out correctly: both derivatives $\partial n/\partial T$ are introduced. The authors of Refs [9, 10] use $(\partial n/\partial T)_{\sigma}$ and (3b); expression (15b) was obtained in [10], and it directly follows from the formulae of [9]. Mit'kin and Shchhavelev [11] also use expression (3b) for a thin glass disk, but do not spec-

ify $\partial n/\partial T$; expression (15a) can be obtained from the formulae given in [11]. Joiner et al. [18, 19], due to the specifics of the problem (their result does not depend on $\partial n/\partial T$, see Section 5.2.1), use (3a) for a rod and (3b) for a disk.

In [4], devoted to Nd:YAG rods in the [111] orientation, the concepts of thermal strain (\mathbf{u}) and elastic strain ($\mathbf{u}' = \mathbf{u} - \alpha T$) are distinguished. Hooke's law (6) with these rather unusual notations does not contain the temperature term and has the form of (A1.1) up to the replacement of \mathbf{u} by \mathbf{u}' . Foster and Osterink [4] describe the photoelastic effect by the expression $\Delta \mathbf{B} = \mathbf{p}\mathbf{u}'$, which, taking into account (4) and the modified formula (A1.1), is equivalent to (3b) up to the constant T_0 not used by them, but leads to more cumbersome expressions for the phase incursion, from which follows (18). For $\partial n/\partial T$, an ambiguous characteristic 'not including any strain effects' is used.

Snitzer [8] used formula (3a) in his review. The value of $\partial n/\partial T$ is not specified, Hooke's law is written in the form of (A1.1), and the tensor \mathbf{u} , in contrast to [4], is called simply strain without any explanation. Despite this, the expressions for phase incursions both in a glass rod and in a glass disk correspond to formula (18). The formulae for the radial dependence of the tensor \mathbf{u}' , identical to those given in [4], are used for Nd:YAG rods by Koechner [12], who also borrowed the value of $\partial n/\partial T$ measured in [4]; the tensor is also simply called the strain. The same formulae are used by him and his coauthor in [3]; moreover, they also call the tensor the elastic strain in the appendix to their work and simply the strain in its main part. The formulae for the refractive indices of the eigenwaves in the rods (see [3, 4, 12]), which can be reduced to (18), are given in the classical book [1], where the tensor \mathbf{u}' is also called simply the strain, and neither expressions nor Hooke's Law (6) nor (A1.1) are given for it (we reviewed all six editions from 1976 to 2006). The thesis, presented in the appendix to [67] and later repeated in [50] in relation to both the book and the paper of the same author [13], that Hooke's law in book [1] is written incorrectly, namely in the form of (A1.1) with a missing temperature term, is apparently erroneous (Cousins [67] gives specific numbers of formulae supposedly containing errors, but the numbering does not correspond to that adopted in any of the book's editions). However, we believe that the reason for the confusion was the inaccuracy of the author of the book in dealing with definitions of physical quantities.

Soms and Tarasov [16] use $(\partial n/\partial T)_\sigma$, but the expressions for P both in a rod and in a disk of a cubic crystal are given in the form of (14), which is obtained using formula (3a). Mezenov et al. [2] give a physical model that describes the temperature change in the refractive index in a medium in the isotropic approximation. This model contains both the term related to a temperature shift of the resonant frequencies of the medium [and representing the contribution of $(\partial n/\partial T)_u$], as well as the terms depending on the density and proportional to the thermal expansion coefficient, i.e., forming $\Delta \chi_T$. Then, when describing the photoelastic effect, an incorrect assertion is made that expression (3b) describes the dependence of the indicatrix of the refractive index on density and therefore generalises the corresponding term in the considered model. (The fallacy of this statement follows, for example, from a thought experiment in which a uniformly heated body is compressed from all sides by external forces so that its dimensions and, therefore, the density are equal to the initial ones. In this case, the stress field, and therefore tensor (3b), in such a body, obviously, will be nonzero.) As a result, the

expression for the increment of the refractive index has adjacent $(\partial n/\partial T)_u$ and (3b) [2]. Then (3b) is replaced by (3a) [obviously, using the incorrect formula (A1.1)], as a result of which the formula by chance again becomes true and corresponding to expression (10a). For the thermo-optical constant P , both expressions (14b) and (15b) are given; they have the same values of $\partial n/\partial T$ and, therefore, contradict each other.

Parfenov et al. [39] use (3b) and do not specify the value of $\partial n/\partial T$. A number of recent works [30, 32, 68] use (3a) and do not specify $\partial n/\partial T$ as well. In [22, 27], (3a) is also used, but the thermal lens is not studied. In their review, Chenais et al. [50] first determine the change in the refractive index through the combination of (3a) with $(\partial n/\partial T)_u$. Then they notice that $(\partial n/\partial T)_u$ is much more difficult to measure than $(\partial n/\partial T)_\sigma$, and they construct a model of a temperature change in the refractive index, similar to that given in [2], but this model also requires a number of assumptions about the symmetry of the medium and, as the authors note, it is not possible to reduce the difference between the two values of $\partial n/\partial T$ to easily measurable physical quantities. We believe that both these models in applied problems of thermoelasticity do not have obvious advantages over the phenomenological approach proposed in [46, 47] [see (7) and (11)]; moreover, Dement'ev [46] argues that calculations using the model from Ref. [50] are inadequate for YAG.

It should also be noted that of all the above works, $\partial n/\partial T$ has the form of a partial derivative only in [7, 9, 16, 39, 46, 47, 50, 67]; in the remaining papers the full derivative is written, which additionally masks the problem of the right choice of the form of writing of photoelastic terms.

Appendix 2

In this appendix, we present the expressions found in [36] for the coefficients \tilde{a}_s , \tilde{b}_s , \tilde{c}_s , and \tilde{d}_{sj} of formula (31). Similarly to this paper, we introduce matrices obtained from elastic compliance and thermal expansion tensors (these quantities are not tensors):

$$\begin{aligned} n_{ijkl} &= s_{ijkl} - s_{zzzz}^{-1} s_{ijzz} s_{zzkl}, \\ \gamma_{Tij} &= \alpha_{Tij} - s_{zzzz}^{-1} s_{ijzz} \alpha_{Tzz}, \end{aligned} \quad (\text{A2.1})$$

where i, j, k and l take the values x, y and z , and denote

$$\begin{aligned} K &= 2n_{xzyz}, \\ K_x &= 3n_{yzyz} + n_{xzxz}, \\ K_y &= n_{yzyz} + 3n_{xzxz}, \\ Q_x &= 3n_{xxzx} + 2n_{xyyz} + n_{yyxz}, \\ Q_y &= 3n_{yyyz} + 2n_{xyxz} + n_{xxyx}, \\ \Lambda &= K_x K_y - K^2, \\ D &= 3n_{xxxx} + 3n_{yyyy} + 2n_{xxyy} + 4n_{xyxy}. \end{aligned} \quad (\text{A2.2})$$

Then

$$\tilde{a}_s = 4 \frac{\Delta(\gamma_{Txx} + \gamma_{Tyy}) + (KQ_y - K_x Q_x) \gamma_{Tzx} + (KQ_x - K_y Q_y) \gamma_{Tyz}}{D\Delta + 2KQ_x Q_y - K_x Q_x^2 - K_y Q_y^2},$$

$$\begin{aligned}
\tilde{b}_s &= \Delta^{-1}[(KQ_x - K_y Q_y)\tilde{a}_s - 4K\gamma_{T,xz} + 4K_y\gamma_{T,yz}], \\
\tilde{c}_s &= \Delta^{-1}[(KQ_y - K_x Q_x)\tilde{a}_s - 4K\gamma_{T,yz} + 4K_x\gamma_{T,xz}], \\
\tilde{d}_{s1} &= 2s_{zzzz}^{-1}[2\alpha_{Tzz} - \tilde{a}_s(s_{xxzz} + s_{yyzz}) - \tilde{b}_s s_{yzzz} - \tilde{c}_s s_{xzzz}], \\
\tilde{d}_{s2} &= 2s_{zzzz}^{-1}[\tilde{a}_s(s_{xxzz} - s_{yyzz}) - \tilde{b}_s s_{yzzz} + \tilde{c}_s s_{xzzz}], \\
\tilde{d}_{s3} &= 2s_{zzzz}^{-1}[2\tilde{a}_s s_{xyzz} + \tilde{b}_s s_{xzzz} + \tilde{c}_s s_{yzzz}].
\end{aligned} \tag{A2.3}$$

Unfortunately, in the particular case of a cubic crystal, expressions (A2.1)–(A2.3) cannot be significantly simplified for arbitrary orientation, despite the significant simplification of the material tensors in the crystallographic coordinate system. In the orientations $[M0N]$ ($\alpha = 0$) and $[MMN]$ ($\alpha = \pi/4$), the tensor s in the coordinate system (x'', y'', z) has a simplified form, due to which

$$\begin{aligned}
K &= 0, \quad \Delta = K_x K_y, \\
Q_y &= 0, \quad \gamma_{T,yz} = 0,
\end{aligned} \tag{A2.4}$$

and (A2.3) can be reduced: in particular, taking into account the change in the notations in (42), (43) and (45)

$$D_3 = D_b = 0. \tag{A2.5}$$

However, the solution still remains cumbersome for an analytical analysis of its dependence on orientation.

In the simplest orientations, i.e. $[001]$ ($\alpha = 0, \beta = 0$), $[011]$ ($\alpha = \pi/2, \beta = -\pi/4$) and $[111]$ ($\alpha = \pi/4, \tan^2\beta = 2$), the coefficients in (A2.3) are reduced to the form

$$\begin{aligned}
D_3 &= D_b = D_c = 0, \\
a_{s[001]} &= \left[(s_{11} + s_{12}) + \frac{s_{11}(\xi_s - 1)}{4} \right]^{-1}, \\
D_{1[001]} &= \frac{3 + \xi_s}{4}, \\
D_{2[001]} &= 0, \\
a_{s[011]} &= \frac{4(1 + 3\xi_s)}{(15\xi_s + 1)s_{11} + 8(\xi_s + 1)s_{12} + 2(\xi_s - 1)^2(s_{11} - s_{12})}, \\
D_{1[011]} &= 1 + \frac{1 - \xi_s}{1 + 3\xi_s} \frac{s_{12} + s_{66}/2}{2s_{11} + 2s_{12} + s_{66}}, \\
D_{2[011]} &= \frac{(1 - \xi_s)(s_{11} - s_{12})}{2s_{11} + 2s_{12} + s_{66}}, \\
a_{s[111]} &= \frac{18\xi_s}{4\xi_s^2(s_{11} - s_{12}) + \xi_s(13s_{11} + 20s_{12}) + s_{11} + 2s_{12}}, \\
D_{1[111]} &= (5 + \xi_s^{-1})/6, \\
D_{2[111]} &= 0
\end{aligned} \tag{A2.6}$$

(the expressions for a_s, D_b and D_c were obtained directly from the formulae of [36]). Note that the coefficient $D_{2[011]}$ can appear in expressions (65) and (70) with a different sign: in the $[101]$ orientation ($\alpha = 0, \beta = -\pi/4$ or $\alpha = \pi, \beta = \pi/4$) $D_2 = D_{2[011]}$, and in the $[110]$ orientation $[110]$ ($\alpha = \pi/4, \beta = -\pi/2$ or $\alpha = 5\pi/4, \beta = \pi/2$) $D_2 = -D_{2[011]}$.

Appendix 3

Here are the coefficients used in formulae (41), (51), (63), (65) and (67):

$$F_1(\xi) = (1 - \xi)f_1 = (1 - \xi)(a_3 + b_3 - 1),$$

$$F_2(\xi, \xi') = (1 - \xi)a_1 - \xi'a_2, \tag{A3.1}$$

$$F_3(\xi, \xi') = (1 - \xi)b_1 - \xi'b_2,$$

$$A_{i\Phi} = A_i \cos 2\Phi + B_i \sin 2\Phi, \tag{A3.2}$$

$$B_{i\Phi} = B_i \cos 2\Phi - A_i \sin 2\Phi,$$

$$A_1(\xi, \xi') = (1 - \xi)a_1 + \xi'a_2, \quad B_1(\xi, \xi') = (1 - \xi)b_1 + \xi'b_2,$$

$$A_2(\xi) = \xi + (1 - \xi)a_3, \quad B_2(\xi, \xi') = (1 - \xi)c_1 + \xi'c_2, \tag{A3.3}$$

$$A_3(\xi, \xi') = (1 - \xi)c_1 - \xi'c_2, \quad B_3(\xi) = \xi + (1 - \xi)b_3,$$

the coefficients with an additional subscript 's' being calculated at $\xi = \xi_s, \xi' = 0$, and the coefficients without this subscript, at $\xi = \xi_\pi, \xi' = \xi_d$. In these expressions

$$a_1 = \frac{1}{4}[\sin^2 2\alpha(1 - \cos^4 \beta) - \sin^2 2\beta],$$

$$a_2 = \frac{1}{2} \cos 2\alpha \cos 2\beta,$$

$$a_3 = \left(1 - \frac{1}{4} \sin^2 2\alpha\right) \sin^4 \beta + \cos^2 2\alpha \cos^2 \beta,$$

$$b_1 = \frac{1}{8} \sin \beta \sin 2\beta \sin 4\alpha,$$

$$b_2 = \frac{1}{4} \sin 2\alpha \cos \beta (1 - 3 \cos^2 \beta),$$

$$b_3 = \sin^2 2\alpha \cos^2 \beta,$$

$$c_1 = -\frac{1}{4} \sin 4\alpha \cos \beta (1 + \cos^2 \beta),$$

$$c_2 = \frac{3}{4} \sin 2\alpha \cos \beta \sin^2 \beta, \tag{A3.4}$$

and f_1 is introduced in (32). Note that

$$\begin{aligned}
A_{1s} &= F_{2s}, \\
B_{1s} &= F_{3s}.
\end{aligned} \tag{A3.5}$$

To calculate thermally induced distortions of a beam in an elastically anisotropic long rod [see (65)], we also need the coefficients

$$\begin{aligned}
 F_4 &= [-(1 - \xi_\pi)b_1 + \xi_d b_2] \tan \beta, \\
 F_5 &= -(1 - \xi_\pi)c_4 + \xi_d c_5, \\
 A_4 &= [-(1 - \xi_\pi)c_1 + \xi_d c_2] \tan \beta, \\
 A_5 &= (1 - \xi_\pi)c_3 - 3\xi_d c_5, \\
 B_4 &= -(1 - \xi_\pi)b_3 \tan \beta, \\
 B_5 &= 2[(1 - \xi_\pi)b_1 + \xi_d c_2] \cot \beta,
 \end{aligned} \tag{A3.6}$$

where

$$\begin{aligned}
 c_3 &= \frac{1}{4}[\sin^2 2\alpha(1 + \cos^2 \beta) \sin 2\beta - \sin 4\beta], \\
 c_4 &= \frac{1}{4}(\sin^2 2\alpha \sin^2 \beta \sin 2\beta + \sin 4\beta), \\
 c_5 &= \frac{\cos 2\alpha \sin 2\beta}{4}.
 \end{aligned} \tag{A3.7}$$

Note that, despite the presence of tangent and cotangent, functions (A3.6) have no singularities at $\beta = i\pi/2$ (where i is an integer) due to the fact that the corresponding coefficients b_j and c_j vanish.

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