Quantum Electronics **50** (4) 392–400 (2020)

https://doi.org/10.1070/QEL17306

Dependence of emittance on the length of an electron bunch during laser-plasma acceleration in guiding structures

M.E. Veisman, N.E. Andreev

Abstract. **We report a theoretical analysis and numerical simulation of the dynamics of transverse emittance of an electron bunch during its acceleration in wake fields generated by a laser pulse in a weakly nonlinear mode. Analytical expressions are obtained for the main factors affecting the emittance growth during acceleration and the case is considered when the characteristic transverse size of the injected bunch exceeds the matched radius determined by the focusing force at the injection point, the initial emittance and the electron bunch energy, and the resulting value of emittance is much larger than the initial one. The dynamics of the emittance growth during acceleration as a function of the length of the electron bunch is described, and the length of the bunch is found at which there occurs a complete phase mixing of betatron oscillations of electrons and the emittance increases to its maximum value determined by the bunch parameters and the focusing force at the injection point. Analytical expressions are in good agreement with the results of numerical simulation.**

Keywords: *laser-plasma electron acceleration, wake fields, electron beam emittance, betatron oscillations.*

1. Introduction

Laser-plasma electron acceleration in guiding structures (in a plasma channel or in a capillary), based on the interaction of electrons with fast electromagnetic waves excited in a plasma by a short intense laser pulse, has attracted the attention of many researchers in recent decades as a promising direction for designing compact sources of accelerated electron bunches. It has been shown both theoretically [1, 2] and experimentally [3] that at a relatively short (~10 cm) capillary length, a propagating laser pulse of sufficient power can capture and form a bunch from background electrons, which will then be accelerated to an energy of ~8 GeV in the wake field of the laser pulse.

However, even such an impressive result today is still insufficient to satisfy the needs of high-energy physics, which requires sources of high-quality (compact, monoenergetic, with a sufficiently large charge) electron bunches with an energy of 1 TeV, i.e. two orders of magnitude higher than the experimentally obtained values, to study the fundamental properties of matter. One way to solve this problem is to use multicascade acceleration schemes in accelerators that boost electrons to such high energies, when a bunch of electrons is accelerated many times, passing one acceleration cascade after another, each of which giving an increment of electron energy by ~10 GeV. The possibility of multistage acceleration has already been experimentally confirmed [4].

When using a multistage accelerator, there arise a number of new problems, for example, the need to ensure fairly accurate matching of cascades with each other in time, the efficiency of electron bunch transfer from cascade to cascade without loss of bunch quality [5, 6], which should be solved in the future for such acceleration schemes to be put into practice. One of the most important problems that must be solved when developing an accelerator with a large number (up to one hundred) of accelerating cascades is to maintain the minimum emittance of an electron bunch during its acceleration in each cascade. Indeed, the emittance of an electron bunch determines the angular and spatial spread of its particles, and the large emittance of an electron bunch in the space between the cascades means a large spread in the electron scattering angle, so that even when use is made of additional focusing by active plasma lenses [4, 7], the entire bunch cannot reach the next acceleration cascade [8].

To solve this problem, it is necessary to determine the main factors affecting the emittance of accelerated electron bunches and analyse the emittance dynamics in the accelerator cascade. The emittance of an electron bunch at the end of the acceleration stage is specified by both its initial value and phase mixing of the electrons in different sections of the bunch under a nonuniform focusing force of the wake fields generated by a laser pulse in a weakly nonlinear mode [9]. Under the assumption of complete phase mixing of the bunch electrons, by calculating the envelope for different phase ellipses [10] in different cross sections (slices) of the electron bunch, Mehrling et al. [11] determined the normalised emittance of the entire bunch at the end of the acceleration stage.

Apart from the growth of the normalised emittance of an electron bunch due to phase mixing of particles, an increase in emittance can also be caused by the nonlinearity of transverse focusing forces, the development of various instabilities and dissipative processes associated with electron acceleration [10]. In addition, a growth of emittance is possible, which is determined by the delay of a group of particles with high transverse velocities (for example, injected away from the axis of the accelerating structure), compared with a group of particles with lower transverse velocities, which is due to the

M.E. Veisman Joint Institute for High Temperatures, Russian Academy of Sciences, ul. Izhorskaya 13, stroenie 2, 125412 Moscow, Russia; e-mail: bme@ihed.ras.ru;
N.E. Andreev Joint Institute for High Temperatures, Russian Academy of Sciences, ul. Izhorskaya 13, stroenie 2, 125412 Moscow, Russia; Moscow Institute of Physics and Technology, Institutskii per. 9, 141701 Dolgoprudnyi, Moscow region, Russia

Received 26 February 2020
Kvantovaya Elektronika **50** (4) 392–400 (2020)
Translated by I.A. Ulitkin

limitation of the total velocity of particles by the speed of light. Such a delay, in particular, was analysed in recent work [12], in which the energy spread of accelerated electrons associated with this delay was calculated. Apart from the reasons mentioned above, a growth of the emittance can result from the violation of cylindrical symmetry due to inaccurate focusing of the electron beam along the axis of the plasma channel or inaccurate focusing of the laser radiation generating the wake field into the plasma channel. This growth was studied by Thévenet at el. [13], and before that by Veisman et al. [14, 8], who also considered the emittance growth due to the misalignment of the accelerated electron beam and the capillary waveguide in which it is accelerated.

In this work, we study the dynamics of the transverse emittance of an electron bunch accelerated in wake fields that are generated by a laser pulse in a weakly nonlinear mode. It is assumed that the laser pulse and the generated wake fields as well as the electron bunch are cylindrically symmetric and propagate collinearly (without displacement of the axes). We neglect the influence of the bunch charge on the generated wake fields and, as a result, on the acceleration of the bunch itself, which is valid at least for bunches with charges not exceeding several picocoulons for the weakly nonlinear regime in question [15]. In addition, we neglect the loss of electron energy stemming from synchrotron radiation during their betatron oscillations, as well as the possible emittance growth caused by this loss. This loss is proportional to the square of the gamma factor of electrons [16] and for the weakly nonlinear regime become noticeable only at accelerated bunch energies exceeding 1 TeV [17].

In solving the problem under consideration, it is assumed that the size of the bunch in the longitudinal direction is sufficiently small (significantly less than the plasma wavelength), since it is known (see, for example, [18–21] that only bunches with a length that is much shorter than the wavelength of the plasma accelerating field can be accelerated in it more or less monoenergetically, which is the necessary prerequisite for an accelerated bunch in many applications. The transverse size of the electron bunch is also assumed to be quite small, much smaller than the characteristic transverse size of the nonuniform wake field, which is an order of the laser spot size. This allows one to use the approximation of the transverse focusing force linearity and to neglect the effect of transverse oscillations of electrons on their longitudinal acceleration.

The case is considered when the initial characteristic transverse size of the injected bunch $r_{\rm b}(t=0)$ exceeds the matched radius $r_{\rm bm} = \sqrt{\epsilon_{\rm n0}/(\gamma_{\rm e} k_\beta)}$ determined by the initial normalised emittance $\epsilon_{\rm n0}$, the electron bunch gamma factor $\gamma_{\rm e}$ and the focusing force at the injection point, which defines the modulus wave vector of betatron oscillation k_β [10, 16]. According to the above expression for the radius $r_{\rm bm}$, the condition $r_{\rm b}(t=0) > r_{\rm bm}$ will be most relevant for relatively small initial values of the emittance $\epsilon_{\rm n0}$ of electron bunches at the entrance to an accelerator cascade and for large gamma factors of bunches $\gamma_{\rm e}$. In this case, the emittance can markedly increase during acceleration, which allows one to neglect its initial value. An increase in emittance results from the phase mixing of the electrons due to the nonuniformity of the focusing force along the length of the accelerated electron bunch and, accordingly, due to a change in the frequency and phase of betatron oscillations in different cross sections of the bunch. In this case, in contrast to [11], where the emittance was determined at the end of acceleration as a result of complete phase mixing, we obtained analytical expressions for a change in the emittance during acceleration as a function of the bunch length and the parameters of accelerating and focusing wake fields, as well as found the conditions under which complete phase mixing takes place, leading to a maximum increase in the emittance of a bunch of accelerated electrons.

This work is aimed at a theoretical analysis of the emittance dynamics of electron bunches during their laser-plasma acceleration in wake fields. Section 2 presents the basic equations. An analytical model using the adiabatic approximation to solve the equations of electron motion is formulated in Section 3. The results of numerical simulation of the emittance dynamics using the initial equations and comparing them with the obtained analytical dependences are presented in Section 4.

2. Basic equations

Let us consider a bunch of electrons accelerated in a wake plasma wave. We assume that both the electron bunch and the wake wave are cylindrically symmetric with respect to their common propagation axis z.

The quantity characterising the transverse spatial distribution of electrons in the bunch and their angular divergence is the transverse emittance, which can be described using its components ϵ_x, ϵ_y along the x, y axes in a plane perpendicular to the electron bunch propagation axis z. According to [10, 11], for the component ϵ_x, the expressions

$$\epsilon_x = \sqrt{\overline{x^2}\,\overline{x'^2} - \overline{xx'}^2}, \quad (1)$$

are valid, where

$$X = \{x, x', xx'\}; \quad \overline{X^2} = N_{\rm b}^{-1}\sum_i (X_i - \bar{X})^2;$$

$$\bar{X} = N_{\rm b}^{-1}\sum_i X_i; \quad x' = {\rm d}x/{\rm d}z = \dot{x}/\dot{z} = p_x/p_z,$$

Here, x_i and x'_i are the coordinate and slope of the trajectory of the ith electron of the bunch, respectively; $N_{\rm b}$ is the number of electrons in the bunch; p_{xi} is the x-component of the momentum of the ith electron, normalised to mc; m is the rest mass of the electron; c is the speed of light; and $\dot{x}$ and $\dot{z}$ are the time derivatives. Formula (1) with the replacement $x \to y$ gives an expression for the component ϵ_y, which in the case of a cylindrically symmetric electron bunch accelerated in cylindrically symmetric fields coincides with the component ϵ_x.

To compensate for the adiabatic decrease of emittance with increasing bunch energy during acceleration, we introduce normalised transverse emittance [10, 22]

$$\epsilon_{x{\rm n}} = \bar{\gamma}_{\rm e}\bar{\beta}_{\rm e}\epsilon_x, \quad (2)$$

where

$$\bar{\gamma}_{\rm e} = \sum_i \gamma_{{\rm e}i}/N_{\rm b}$$

is the gamma factor of the bunch;

$$\gamma_{{\rm e}i} = \sqrt{1 + p_{xi}^2 + p_{yi}^2 + p_{zi}^2}$$

is the gamma factor of the ith electron; and

$$\bar{\beta}_{\rm e} = \sqrt{1 - \bar{\gamma}_e^{-2}} \approx 1$$

is the beta factor for the bunch of relativistic electrons discussed below.

The transverse normalised emittance for the entire xy plane can be described by the expression

$$\epsilon_{\mathrm{n}} = \sqrt{2(\epsilon_{x\mathrm{n}}^2 + \epsilon_{y\mathrm{n}}^2)}, \tag{3}$$

with $\epsilon_{\mathrm{n}} = 2\epsilon_{x\mathrm{n}} = 2\epsilon_{y\mathrm{n}}$ for the cylindrically symmetric case investigated below.

To determine the coordinates, momenta and gamma factor of the ith electron of the accelerated bunch, we need to solve the relativistic equations of motion in the wake wave fields [23, 24]. For cylindrically symmetric wake fields, these equations in Cartesian coordinates have the form (the subscript i for the ith electron is omitted hereinafter for brevity)

$$\mathrm{d}p_z/\mathrm{d}\tau = \partial_\xi \phi, \tag{4}$$

$$\mathrm{d}\xi/\mathrm{d}\tau = \gamma_{\mathrm{e}}^{-1} p_z - 1, \tag{5}$$

$$\mathrm{d}p_x/\mathrm{d}\tau = (\tilde{x}/\rho)\partial_\rho \phi, \quad \mathrm{d}p_y/\mathrm{d}\tau = (\tilde{y}/\rho)\partial_\rho \phi, \tag{6}$$

$$\mathrm{d}\tilde{x}/\mathrm{d}\tau = \gamma_{\mathrm{e}}^{-1} p_x, \quad \mathrm{d}\tilde{y}/\mathrm{d}\tau = \gamma_{\mathrm{e}}^{-1} p_y, \tag{7}$$

where $\phi = \phi(\xi,\rho,\tau)$ is the wake potential normalised to mc^2/e [25]. Hereinafter, we use the dimensionless time $\tau = \omega_{\mathrm{p}} t$ and coordinates $\xi = k_{\mathrm{p}}(z - ct)$, $\rho = \sqrt{\tilde{x}^2 + \tilde{y}^2}$, $\tilde{x} = k_{\mathrm{p}} x$, $\tilde{y} = k_{\mathrm{p}} y$, where $k_{\mathrm{p}} = \omega_{\mathrm{p}}/c$; $\omega_{\mathrm{p}} = \sqrt{4\pi e^2 n_{\mathrm{e}}/m}$ is the plasma frequency; n_{e} is the concentration of background electrons in the plasma; and e is the electron charge.

3. Analytical model

If the radius of the accelerated electron bunch is much smaller than the characteristic transverse scale of the wake field, then in expanding the potential of the wake field near the axis, we can confine ourselves to the quadratic term, which corresponds to a linear increase in the radial focusing force with increasing distance from the axis. Moreover, according to equations (6) and (7), the electron trajectory in the transverse plane xy is determined by the equations:

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tau^2} + \frac{\mathrm{d}(\ln\gamma_{\mathrm{e}})}{\mathrm{d}\tau}\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tau} + \Omega^2\tilde{x} = 0, \tag{8}$$

$$\Omega(\xi,\tau) = \sqrt{\alpha(\xi,\tau)/\gamma_{\mathrm{e}}(\xi,\rho,\tau)}, \quad \alpha(\xi,\tau) = -\rho^{-1}\partial_\rho\phi. \tag{9}$$

The equation for the y coordinate coincides with equation (8) when a replacement $\tilde{x} \to \tilde{y}$ is made. These equations describe the betatron oscillations of electrons in the transverse plane xy, the frequency of which Ω (9) is different in different sections of the bunch and changes with time with increasing gamma factor as the electron gains energy during acceleration and with increasing focusing force when the wake wave phase 'lags' behind an ultrarelativistic electron moving at a velocity that practically coincides with the speed of light.

Equations (8) for $\tilde{x}$ and $\tilde{y}$ coordinates are related to equations describing the longitudinal motion of electrons, which follow from (4) and (5). For ultrarelativistic electrons accelerated along the z axis, when calculating the gamma factor, we can neglect the transverse momentum of the electrons compared to the longitudinal one, the condition for which is the inequality $|p_x/p_z| \approx |\mathrm{d}\tilde{x}/\mathrm{d}\tau| \approx \Omega|\tilde{x}| \ll 1$ and a similar inequality for $|p_y/p_z|$. In this case, we obtain the expressions for the gamma factor and the corresponding coordinate ξ of the electron in question at some point in time τ:

$$\gamma_{\mathrm{e}}(\xi,\rho,\tau) = \gamma_0 + \int_0^\tau \partial_\xi\phi(\xi,\rho,\tau)\,\mathrm{d}\tau, \tag{10}$$

$$\mathrm{d}\xi/\mathrm{d}\tau = -\gamma_{\mathrm{e}}^{-2}(\xi,\rho,\tau)/2, \tag{11}$$

where γ_0 is the initial value of the gamma factor of the electron at the moment of bunch injection. The dependences of γ_{e} and ξ on ρ determine the relationship between the transverse and longitudinal motions of the electron, which are respectively described by equation (8) for the $\tilde{x}$ and $\tilde{y}$ coordinates and equations (10), (11).

Below, we restrict ourselves to the case of sufficiently compact electron bunches, the transverse size of which is small compared to the characteristic scale of changes in the wake field, which allows us to neglect the differences in the electron energies in the given transverse cross section of the bunch, i.e., the influence of the transverse motion on the electron energy gain:

$$\gamma_{\mathrm{e}}(\xi,\rho,\tau) \approx \gamma_{\mathrm{e}}(\xi,\rho = 0,\tau) = \gamma_{\mathrm{e}}(\xi,\tau). \tag{12}$$

Since for ultrarelativistic electrons ($\gamma_{\mathrm{e}} \gg 1$) the difference between the longitudinal velocity and the speed of light is very small [see (11)], we assume that all the electrons of the bunch move at the same velocity equal to the speed of light, which corresponds to a constant value of the accompanying longitudinal coordinate of the electron ξ, equal to its initial value at the time of injection ξ_0, and, accordingly, the constant length of a bunch of accelerated electrons. Moreover, all the coefficients in equation (8) depend only on the time τ and the initial coordinate of the injection $\xi = \xi_0$, and the slope of the electron bunch trajectory is equal to the transverse velocity of the electron, normalised to the speed of light:

$$x' = \mathrm{d}x/\mathrm{d}z = \dot{x}/\dot{z} = \mathrm{d}\tilde{x}/\mathrm{d}\tau.$$

For typical laser-plasma acceleration parameters, the period of betatron oscillations is much shorter than the characteristic time of the change in the betatron frequency and the electron relativistic gamma factor:

$$\Omega^{-1}|\partial\ln\Omega/\partial\tau| \ll 1, \quad \Omega^{-1}|\partial\ln\gamma_{\mathrm{e}}/\partial\tau| \ll 1. \tag{13}$$

Under the conditions corresponding to the adiabatic approximation, the slice emittance in each transverse cross section of the electron bunch is preserved [10, 22, 26] for an arbitrary bunch radius [with a linear dependence of the focusing force on the radius (8), (9)]. However, if the radius of the bunch is not consistent with the values of the emittance and focusing force, the bunch radius will oscillate at a doubled betatron frequency [10, 16, 22]. Due to a change in the betatron frequency along the bunch length (due to the nonuniformity of the focusing force in a plasma wake wave), the electrons in different sections of the bunch oscillate with different phases. In this case, phase mixing leads to an increase in the emittance of the entire bunch, since the region of the phase space (x, p_x) occupied by the electrons of the entire bunch is the combination of the phase space regions occupied by electrons from individual transverse cross sections (slices) and exceeds each of these regions in area [11, 10]. An increase in the emittance of an electron bunch during acceleration can be quite significant if the initial radius of the bunch noticeably exceeds the matched one. This case is considered below, when the transverse momenta of the electrons of the injected beam, determined by the initial emittance, are much smaller than the characteristic transverse momenta acquired by the electrons under the action of the focusing force (8). In this case, the emittance of the entire bunch due to mixing of the phases of the betatron oscillations (for a sufficiently long bunch and acceleration lengths) is much larger than the initial emittance of the injected beam. This allows us to neglect the initial emittance of the bunch and write the solution of equations (8) in the form

$$\tilde{x}(\tilde{x}_0,\xi,\tau) = \tilde{x}_0\varkappa(\xi,\tau), \quad \tilde{y}(\tilde{y}_0,\xi,\tau) = \tilde{y}_0\varkappa(\xi,\tau),$$
$$\varkappa(\xi,\tau) = \sqrt{\frac{\Omega_0(\xi)\gamma_0}{\Omega(\xi,\tau)\gamma_e(\xi,\tau)}}\cos\left[\int_0^\tau \Omega(\xi,\tau')\,d\tau'\right], \tag{14}$$

where $\tilde{x}_0$ and $\tilde{y}_0$ are the initial (at the moment of injection at $\tau = 0$) electron coordinates; $\varkappa$ is the solution to equations (8) with the initial conditions $\varkappa(\tau = 0) = 1$ and $d\varkappa/d\tau(\tau = 0) \approx 0$; $\Omega(\xi,\tau)$ is defined in (9) with $\gamma_e(\xi,\tau)$ (12); $\Omega_0(\xi) = \Omega(\xi,\tau = 0)$; and the dependence of γ_e on ξ is neglected (i.e. the injected bunch is monoenergetic). Expression (14) is written in the adiabatic approximation up to terms of the main order in small parameters (13).

We will assume below that the electron distribution in the injected bunch is Gaussian:

$$n_b(\xi,\rho_0,\tau = 0) = n_\parallel(\xi)n_\perp(\rho_0), \quad \rho_0 = \sqrt{\tilde{x}_0^2 + \tilde{y}_0^2};$$
$$n_\parallel(\xi) = \pi^{-1/2}\sigma_z^{-1}\exp(-\xi^2/\sigma_z^2), \tag{15}$$
$$n_\perp(\rho_0) = \pi^{-1}\sigma_{r0}^{-2}\exp(-\rho_0^2/\sigma_{r0}^2),$$

where the distributions in (15) are normalised to unity; σ_{r0} is the initial dimensionless rms radius of the bunch; and σ_z is its characteristic length. Moreover, for each transverse cross section of the bunch with coordinate ξ we find the cross section average values:

$$\overline{x^2(\xi,\tau)} = \frac{1}{k_p^2}\iint_{-\infty}^{\infty}\tilde{x}^2(\tilde{x}_0,\xi,\tau)n_\perp(\rho_0)\,d\tilde{x}_0 d\tilde{y}_0$$
$$= \frac{1}{2k_p^2}[\sigma_{r0}\varkappa(\xi,\tau)]^2, \tag{16}$$
$$\overline{(dx/d\tau)^2} = \frac{1}{2k_p^2}[\sigma_{r0}d\varkappa(\xi,\tau)/d\tau]^2.$$

Then from (1)–(3) for the normalised transverse emittance we obtain the expression

$$\epsilon_n(\tau) = k_p^{-1}\gamma_e\sigma_{r0}^2\sqrt{\langle\varkappa^2\rangle_\xi\langle(d\varkappa/d\tau)^2\rangle_\xi - \langle\varkappa d\varkappa/d\tau\rangle_\xi^2}, \tag{17}$$

where averaging over the electron bunch length is determined by the formula

$$\langle A\rangle_\xi \equiv \int_{-\infty}^{\infty}A(\xi)n_\parallel(\xi)\,d\xi. \tag{18}$$

It follows from (14), (16) and (17) that the temporal change of the emittance and rms radius of the bunch is described by the expressions

$$\epsilon_n(\tau) = \frac{1}{2k_p^2}\sigma_{r0}^2\gamma_0\Omega_0[(1 + \langle\cos\Psi\rangle_\xi)(1 - \langle\cos\Psi\rangle_\xi)$$
$$- \langle\sin\Psi\rangle_\xi^2]^{1/2}, \tag{19}$$

$$r_b(\tau) = \sqrt{\langle\overline{x^2} + \overline{y^2}\rangle_\xi} = \frac{r_{b0}}{\sqrt{2}}\left[\frac{\gamma_0\Omega_0}{\gamma(\tau)\Omega(\tau)}(1 + \langle\cos\Psi\rangle_\xi)\right]^{1/2},$$
$$r_{b0} = \sigma_{r0}/k_p, \tag{20}$$

where

$$\Psi = \Psi(\xi,\tau) = 2\int_0^\tau \Omega(\xi,\tau')\,d\tau'. \tag{21}$$

Since for practically important cases of acceleration of electron bunches with a small energy spread, their length should be much shorter than the wake plasma wavelength, in deriving expressions (19) and (20) we take into account changes in the focusing force and electron energy along the bunch length only in oscillating terms containing Ψ. In the remaining factors, both the betatron frequency and the gamma factor are determined by their values in the bunch centre: $\Omega(\tau) = \Omega(\xi_c,\tau)$, $\gamma(\tau) = \gamma(\xi_c,\tau)$, where ξ_c is the coordinate of the bunch centre.

It follows from expressions (19)–(21) that at a constant betatron frequency along the bunch length [i.e., at Ω and, accordingly, the phase of betatron oscillations (21), independent of ξ], the emittance of the entire bunch coincides with the slice emittance and, therefore, is preserved, remaining equal to the initial value, taken as zero. The radius of the bunch oscillates at a doubled betatron frequency from the maximum value to zero. With increasing focusing force and electron energy, this maximum value adiabatically decreases.

The nonuniformity of the focusing force along the bunch length, which causes phase mixing of the electron trajectories, leads to an increase in emittance during acceleration up to an asymptotic value

$$\hat{\epsilon}_n = \frac{1}{2}k_p r_{b0}^2\gamma_0\Omega_0, \tag{22}$$

corresponding to complete mixing, when the phase difference of betatron oscillations along the bunch length becomes much greater than π, i.e.

$$\sigma_z|\partial\Psi(\xi,\tau)/\partial\xi|_{\xi=\xi_c} \gg \pi, \tag{23}$$

and $\langle\cos\Psi\rangle_\xi \approx \langle\sin\Psi\rangle_\xi \approx 0$. Obviously, a necessary (but not sufficient) condition for inequality (23) to hold is the requirement that the considered acceleration time τ be much longer than the period of betatron oscillations of electrons in the bunch centre at $\xi = \xi_c$. The radius of the electron bunch when inequality (23) is satisfied, approaches, according to (20), the value of

$$\hat{r}_b(\tau) = \frac{r_{b0}}{\sqrt{2}}\left[\frac{\gamma_0\Omega_0}{\gamma(\tau)\Omega(\tau)}\right]^{1/2} \tag{24}$$

slowly decreasing with increasing energy and focusing force. This expression for the average radius of the electron bunch is equal to the radius r_{bm}, which is matched with the constant asymptotic value of emittance (22), as well as with the energy of the bunch electrons and the focusing force acting at a given moment of time [16]:

$$\hat{r}_b(\tau) = r_{bm}(\tau) = \left[\frac{\hat{\epsilon}_n}{k_p\gamma(\tau)\Omega(\tau)}\right]^{1/2} \equiv \left[\frac{\hat{\epsilon}_n}{\gamma(\tau)k_\beta(\tau)}\right]^{1/2}, \quad (25)$$

where $k_\beta(\tau) = k_p\Omega(\tau)$ is the wavenumber of betatron oscillations. Therefore, after reaching the values of (22) as a result of mixing of the phases of the betatron oscillations of the bunch electrons, the emittance remains constant.

To determine the emittance dynamics and specific conditions for achieving the asymptotic value (22), we consider the acceleration of an electron bunch in a quasi-stationary wake wave generated by a short intense laser pulse in a matched plasma channel with a given parabolic radial plasma concentration profile:

$$n_e(r) = n_{e0}[1 + (r/R_{ch})^2], \quad (26)$$

where n_{e0} is the plasma concentration on the channel axis, and R_{ch} is the channel radius. For a Gaussian radial distribution of the laser pulse field, the condition for matching the channel radius with the size of the focal spot of the laser radiation is determined by the equality $R_{ch} = k_p r_0^2/2$, where r_0 is the exponential (in the laser field amplitude) transverse half-width of the laser spot, and the wavenumber of the plasma wave $k_p = \omega_p/c$ is determined by the electron concentration n_{e0} on the channel axis [25, 27–30]. Moreover, if the laser pulse power does not exceed the critical self-focusing power, the laser pulse propagates in a channel with an almost constant amplitude and generates a quasi-stationary wake wave propagating with a phase velocity determined by the group velocity of the laser pulse, so that for the phase wave velocity v_{ph}, the relativistic gamma factor $\gamma_{ph} = (1 - \beta_{ph}^2)^{-1/2} \approx k_0/k_p$, where $\beta_{ph} = v_{ph}/c$, $k_0 = \omega_0/c$ and ω_0 is the laser frequency.

The potential of the wake field generated by a laser pulse with a focal spot size exceeding the plasma wavelength ($r_0 > 2\pi/k_p$) can be represented in the approximation linearised by the wake wave amplitude as

$$\phi(\xi,\rho,\tau) = \phi_0\exp(-\rho^2/\rho_0^2)\sin[\xi + (1-\beta_{ph})\tau],$$
$$1-\beta_{ph} \approx 0.5\gamma_{ph}^{-2}, \quad (27)$$

where $\rho_0 = k_p r_0/\sqrt{2}$ is the dimensionless characteristic radius of the region occupied by the plasma wake wave, and the potential amplitude ϕ_0 is determined by the laser pulse intensity and duration [9, 25] in the case when the effect of ionisation on the wake field generation can be neglected [31].

Moreover, according to (9), (10) and (12), we obtain the expressions

$$\alpha(\xi,\tau) = (2\phi_0/\rho_0^2)\sin(\xi + \tilde{\tau}),\ \tilde{\tau} = (1-\beta_{ph})\tau,$$
$$\gamma_e(\xi,\tau) = \gamma_0 + 2\gamma_{ph}^2\phi_0[\sin(\xi + \tilde{\tau}) - \sin\xi]. \quad (28)$$

We assume that an electron bunch is injected into the focusing phase of the wake wave (27) in the vicinity of the maximum of the accelerating force; i.e., at $\xi \geqslant 0$. In this case, bearing in mind that the bunch length is much shorter than the wake wavelength, the expression for the betatron frequency (9) in the region $\xi \ll 1$, where the bunch electrons are located, can be written as

$$\Omega(\xi,\tau) = \left\{\frac{\alpha_{max}(\sin\tilde{\tau} + \xi\cos\tilde{\tau})}{\gamma_0 + 2\gamma_{ph}^2\phi_0[\sin\tilde{\tau} - (1-\cos\tilde{\tau})\xi]}\right\}^{1/2},$$
$$\alpha_{max} = 2\phi_0/\rho_0^2. \quad (29)$$

According to (28), the maximum energy increase in one section of a laser-plasma accelerator (in the weakly nonlinear regime of wake wave excitation) $\Delta\gamma_{max} = 2\gamma_{ph}^2\phi_0$ corresponds to $\tilde{\tau} = \pi/2$, at which the electron, injected at $\tau = 0$ to the maximum of the accelerating force at the focusing phase boundary, reaches a maximum of the potential (27) at $\xi = 0$. The acceleration length corresponding to $\tilde{\tau} = \pi/2$ is equal to half the length of dephasing (at which the electron passes through the entire accelerating phase, i. e. from a minimum potential to a maximum one): $L_{ph} = k_p^{-1}\pi/(1-\beta_{ph}) = \lambda_p\gamma_{ph}^2 = \lambda_0\gamma_{ph}^3$, where $\lambda_0 = 2\pi/k_0$ and $\lambda_p = 2\pi/k_p$ are the wavelengths of the laser radiation and the plasma wave wake, respectively. For electrons injected with an energy exceeding the maximum energy gain ($\gamma_0 \gg \Delta\gamma_{max}$), taking into account $\Omega(\xi,\tau) \approx \Omega_{max}(\sin\tilde{\tau} + \xi\cos\tilde{\tau})^{1/2}$, we obtain the expression for the phase of the betatron oscillations of the bunch electrons (21), decomposed along the short length of the bunch near the position of its centre ξ_c, with a not too short acceleration length, when $\tilde{\tau} > \xi$:

$$\Psi(\xi,\tau) = \Psi(\xi_c,\tau) + 2(\xi - \xi_c)\psi_0(\tau),$$
$$\psi_0(\tau) = 2\gamma_{ph}^2\,\Omega_{max}[\sin^{1/2}(\tilde{\tau} + \xi_c) - \sin^{1/2}\xi_c], \quad (30)$$
$$\Omega_{max} = [2\phi_0/(\rho_0^2\gamma_0)]^{1/2}.$$

Averaging over the bunch length (18) in formulae (19) and (20) yields the expressions

$$\epsilon_n(\tau) = \hat{\epsilon}_n[1 - A^2(\tau,\sigma_z)]^{1/2},$$
$$A^2(\tau,\sigma_z) = \exp\{-2[\psi_0(\tau)\sigma_z]^2\}, \quad (31)$$

$$r_b(\tau) = \hat{r}_b(\tau)\{1 + A(\tau,\sigma_z)\cos[\Psi(\xi_c,\tau)]\}^{1/2}, \quad (32)$$

for the emittance dynamics and the average radius of the bunch as a function of its length σ_z, where the phase $\Psi(\xi_c,\tau)$ is determined by formulae (21) and (29) for the bunch centre at $\xi = \xi_c$. Calculating the corresponding integral in the approximation of low energy gained by the electrons, compared with their initial energy, which is valid in accordance with (28) under the condition $\mu = 2\gamma_{ph}^2\phi_0/\gamma_0 < 1$, and also using the expansion of the term $\sin(\tilde{\tau} + \xi_c)$ in a Taylor series with retention of the first three terms of the series, we can write an approximate expression for $\Psi(\xi_c,\tau)$ in the form

$$\Psi(\xi_c,\tau) = 4\gamma_{ph}^2\Omega_{max}(1-\mu\sin\xi_c)^{-1/2}[I_b(\tilde{\tau}+\xi_c) - I_b(\xi_c)],$$

$$I_b(t) = t^{3/2}\Big(\frac{2}{3} - \frac{b}{5}t + \frac{9b^2-1}{42}t^2\Big), \quad (33)$$

$$b = \frac{\mu}{1-\mu\sin\xi_c}, \quad \mu = \frac{2\gamma_{ph}^2\phi_0}{\gamma_0} \equiv \Delta\gamma_{max}/\gamma_0.$$

Formula (31) shows that in order to increase the emittance to its maximum value (22), determined by the radius of the bunch, its energy and focusing force at the injection point, the exponent in the expression for $A^2(\tau,\sigma_z)$ should become greater than unity over time, for which the condition

$$\sigma_z > [\sqrt{2}\psi_0(\tau)]^{-1} = \{2\sqrt{2}\gamma_{ph}^2\Omega_{max}[\sin^{1/2}(\tilde{\tau}+\xi_c) - \sin^{1/2}\xi_c]\}^{-1} \equiv \tilde{\sigma}_z(\tau) \quad (34)$$

should be met. For the betatron oscillations of electrons to be phase mixed at a maximum acceleration length (at $\tilde{\tau}+\xi_c = \pi/2$), the length of the electron bunch should exceed a minimum value

$$\sigma_{min} = (2\sqrt{2}\gamma_{ph}^2\Omega_{max})^{-1} = \frac{1}{2\sqrt{2}}\frac{k_p}{k_{\beta max}}\gamma_{ph}^{-2}, \quad (35)$$

where $k_{\beta max} = k_p\Omega_{max}$ is the wavenumber corresponding to the maximum frequency of betatron oscillations in (30). For shorter bunches ($\sigma_z < \sigma_{min}$), as well as for shorter acceleration times, when the inequality opposite to (34) holds, the emittance of the bunch increases to a value (31) that is less than the asymptotic one (22):

$$\epsilon_n(\tau) = \hat{\epsilon}_n[\sigma_z/\tilde{\sigma}_z(\tau)]. \quad (36)$$

For bunches whose length exceeds the minimum one (35), inequality (34) can be written as a condition for the acceleration length $z_{acc} = ct = \tau/k_p$ for a given bunch length σ_z depending on the electron energy and the wake field parameters (the maximum focusing force and the wake phase velocity):

$$z_{acc}/L_{ph} > \pi^{-1}\arcsin(\sigma_{min}^2/\sigma_z^2), \quad \sigma_z > \sigma_{min}. \quad (37)$$

Finally, we note that the adiabaticity conditions (13) with allowance for (27) and (28) can be represented in the form of restrictions on the wake potential amplitude, electron energy and electron bunch injection coordinate (phase) ξ_c:

$$\frac{\rho_0}{4\gamma_{ph}^2}(\tilde{\tau}+\xi_c)^{-3/2} \ll \Big(\frac{2\phi_0}{\gamma_0}\Big)^{1/2} \ll \frac{2}{\rho_0}(\tilde{\tau}+\xi_c)^{1/2}. \quad (38)$$

While for typical parameters ($2\phi_0 \leqslant 1$, $\gamma_0 > 10^3$ and $\rho_0 \approx 3$) the right-hand inequality in (38) is satisfied, with the exception of a small time interval when an electron bunch is injected directly onto the boundary of the focusing phase ($\xi_c \to 0$), the left-hand inequality can impose very significant restrictions on the injection coordinate and electron bunch energy.

4. Numerical calculations

The dynamics of the emittance of an electron bunch during its acceleration in wake fields generated by a laser pulse in the weakly nonlinear regime in the plasma channel (26) was simulated by numerically solving electron motion equations (4)–(7) with the wake potential (27). The dimensionless amplitude of the wake potential is $\phi_0 = 0.095$, and its characteristic radius is $\rho_0 = 3.47$, which corresponds to a maximum 'focusing force rigidity' $\alpha_{max} = 0.0158$ in (29) and a maximum electron energy increase $\Delta\gamma_{max} = 1216$. The plasma concentration n_{e0} on the channel axis [see (26)] was chosen such that the determined gamma factor of the plasma wave is $\gamma_{ph} = k_0/k_p = 80$ and the dephasing length is $L_{ph} = 41$ cm. At a laser radiation wavelength of $\lambda_0 = 0.8$ μm, the radius of the matched plasma channel is $R_{ch} = 123$ μm for a characteristic size of the laser focal spot $r_0 = 50$ μm. The amplitude of the potential $\phi_0 = 0.095$ corresponds to resonant excitation of the wake wave by a laser pulse of duration $\tau_{FWHM} = 80$ fs with a dimensionless amplitude $a_0 = eE_{max}/(m\omega_0 c) = 0.5$ [9].

The injected electron bunches had a Gaussian particle distribution in the longitudinal and transverse directions (15) with dimensionless half-widths σ_z and σ_{r0}, respectively: in the calculations, σ_z are different, and $\sigma_{r0} = 0.212$. Electron bunches were injected at various displacements of the position of the electron bunch centre ξ_c relative to the position of the maximum accelerating force. In this case, the transverse normalised emittance was determined by formulae (1)–(3).

We emphasise that in simulating the particle motion by equations (4)–(7) with potential (27), the radial force is assumed to be nonlinear, the dependence of the electron gamma factor γ_e on the radial (ρ) and accompanying (ξ) coordinates is not neglected and adiabaticity conditions (13) are considered unfulfilled.

The simulation results are shown in Figs 1 and 2 (solid curves) in comparison with analytical expressions (31) for the normalised emittance and with (32) for the rms radius of the accelerated electron bunch with an approximate phase value (33) (dashed curves).

In accordance with formulae (30)–(35), at low injection energies $E_{inj} = mc^2\gamma_0$, the value of σ_{min} is quite small, and the values of the phase ψ_0, on the contrary, are quite large. Therefore, under the condition $\sigma_z > \sigma_{min}$, the oscillations of the rms radius of the electron beam rapidly decay and the normalised emittance quickly reaches an asymptotic value (22) (Fig. 1a and the lower curves in Fig. 2) with increasing acceleration length z_{acc}. With an increase in the injection energy (Figs 1c and 1d and Fig. 2), σ_{min} grows and ψ_0 decreases. Under the condition $\sigma_z > \sigma_{min}$, the oscillations of the rms radius of the electron beam decay, but more slowly than in the case of relatively low injection energies (cf. Figs 1a and 1c). Due to the decrease in $\Omega_{max} \propto \gamma_0^{-1/2}$, the phase mixing proceeds more slowly with increasing injection energy, which also manifests itself in a slower increase in the normalised emittance with an increase in the acceleration length, as compared with the case of small γ_0.

For $\sigma_z < \sigma_{min}$, the oscillations of the rms radius of the electron bunch weakly decay with increasing acceleration length. In this case, the normalised emittance at the end of acceleration reaches values that are less than or equal to those in the linear limit (36) (Figs 1b and 1d and Fig. 2). In this case, we observe incomplete phase mixing of electrons initially injected into different phases of the wake wave.

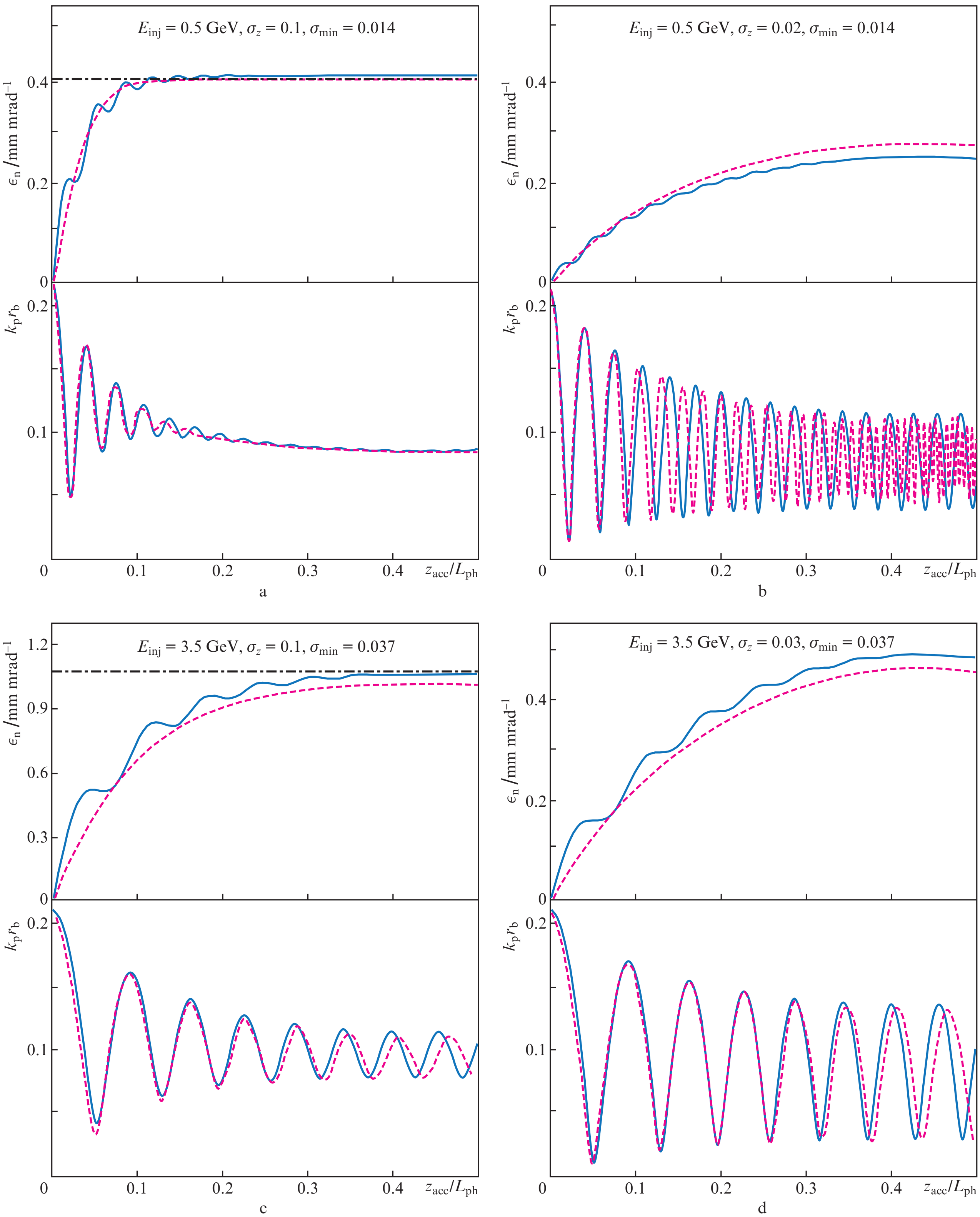

Figure 1. Dependences of the normalised emittance ϵ_n and rms dimensionless radius of the electron bunch $k_p r_b$ on the dimensionless acceleration length z_{acc}/L_{ph} for various injection energies E_{inj}, various dimensionless lengths σ_z and σ_{min} and for the injection point of electron bunches $\xi_c = 0.2$. The dot-and-dash lines show the asymptotic limit. The calculation parameters are presented in the text.

Note that in accordance with formulae (22) and (36), the asymptotic value of the normalised emittance is proportional to $\sqrt{\gamma_0}$, i.e., it increases with increasing injection energy, while in the linear limit, which is realised at $\sigma_z < \sigma_{min}$, the value of the normalised emittance at the instant of time τ is proportional to $\sqrt{\gamma_0/\gamma_e(\tau)}$, i.e., it does not increase with increasing injection energy (see also Fig. 2 for small σ_z). This situation may be favourable for accelerator cascades with high injection energies if it is possible to ensure a sufficiently short length of the accelerated electron bunch.

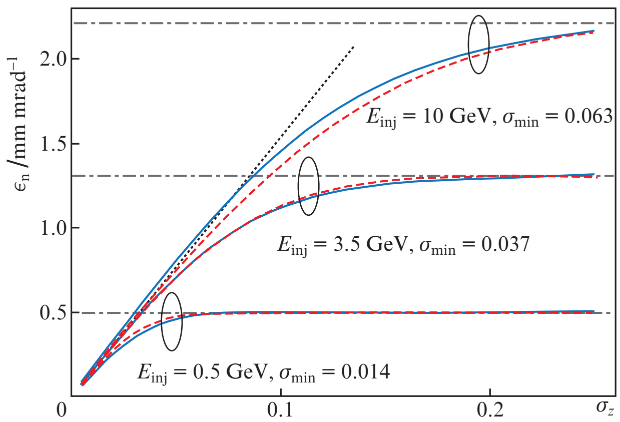

Figure 2. Dependences of the normalised emittance ϵ_n at the end of the acceleration stage (at $z_{acc} = L_{ph}/2$) on the dimensionless length σ_z of electron bunches at various injection energies E_{inj}, different lengths σ_{min} and $\xi_c = 0.3$. The straight dotted line is the linear limit (36), dot-and-dash lines are the asymptotic limits. The calculation parameters are presented in the text.

Results shown in Figs 1 and 2 show that the analytical model is in good agreement with the results of numerical simulation.

5. Conclusions

We have analysed the dynamics of the normalised emittance of a short electron bunch accelerated in the wake plasma fields generated in guiding structure (plasma channel) by an intense laser pulse in the weakly nonlinear regime. For the case, when the characteristic transverse size of the injected bunch exceeds the matched radius determined by the value of the focusing force at the injection point, the initial emittance and the electron bunch energy, and when the initial emittance is much smaller than the final one, we have considered the mechanisms of the normalised emittance growth caused by the phase mixing of electrons due to the focusing force nonuniformity along the length of the accelerated electron bunch and, therefore, due to a change in the frequency and phase of betatron oscillation in different transverse cross section of the bunch.

The obtained analytical expressions (31) and (32) determine the dependences of the normalised emittance and the rms radius of the electron bunch on the acceleration length (time), the parameters of the accelerating and focusing wake fields, and the characteristics of the electron bunch at the injection point: its length, radius, energy and injection phase. We have found the length of the electron bunch (34), (35) at which there occurs complete phase mixing and the emittance increases to its maximum value (22), which is determined by the radius of the bunch, its energy and focusing force at the injection point.

We emphasise that for relatively long electron bunches, the normalised emittance caused by complete phase mixing increases with increasing injection energy (the gamma factor of the bunch γ_0), while this growth is not observed for relatively short bunches. This peculiarity is especially important for multistage laser-plasma accelerators of electron to high (up to several terawatt) energies.

The results of a numerical simulation of particle motion using the equations of relativistic dynamics (4)–(7) (without assuming linearity of the radial force, independence of longitudinal and transverse movements, and without fulfilling the adiabaticity condition) are in good agreement with the obtained analytical expressions (31) and (32) for the dependences of the normalised emittance and the rms radius of the electron bunch on the acceleration length (time) and the parameters of the wake fields and the accelerated bunch (Figs 1 and 2).

A discussion of the dynamics of the emittance $\epsilon_n(t)$ of electron bunches, taking into account its final value $\epsilon_n(t = 0)$ at the time of injection, as well as of a practically important issue of maintaining the emittance values during multistage acceleration, is the subject of our next work.

Acknowledgements. The authors are grateful to S.V. Kuznetsov for valuable comments and discussions.

This work was partially supported by the Presidium of the Russian Academy of Sciences (Fundamental Research Programme ‘Extreme Light Fields and Their Interaction with Matter) and the Russian Foundation for Basic Research (Grant No. 19-02-00908).

References

1. Esarey E., Schroeder C.B., Leemans V.P. *Rev. Mod. Phys.*, **81**, 1229 (2009).
2. Mourou G.A., Tajima T., Bulanov S.V. *Rev. Mod. Phys.*, **78**, 309 (2006).
3. Gonsalves A.J., Nakamura K., Daniels J., et al. *Phys. Rev. Lett.*, **122**, 084801 (2019).
4. Steinke S., van Tilborg J., Benedetti C., et al. *Phys. Plasmas*, **23** (5), 056705 (2016).
5. Tomassini P., Rossi A.R. *Plasma Phys. Controlled Fusion*, **58** (3), 034001 (2016).
6. Chancé A., Li X., Nghiem P. *Proc. 10th Int. Particle Accelerator Conf. (IPAC'19)* (Melbourne, Australia, 2019; Geneva, Svitzerland: JACoW Publishing, 2019) pp 3594–3597; https://doi.org/10.18429/JACoW-IPAC2019-THPGW006.
7. Pompili R., Anania M.P., Bellaveglia M., et al. *Phys. Rev. Lett.*, **121**, 174801 (2019); https://link.aps.org/doi/10.1103/PhysRevLett.121.174801.
8. Veisman M.E., Kuznetsov S.V., Andreev N.E. *Quantum Electron.*, **47** (3), 199 (2017) [*Kvantovaya Elektron.*, **47** (3), 199 (2017)].
9. Gorbunov L.M., Kirsanov V.I. *Sov. Phys. JETP*, **66**, 290 (1987) [*Zh. Eksp. Teor. Fiz.*, **93**, 509 (1987)].
10. Reiser M. *Theory and Design of Charged Particle Beams* (Wiley-VCH Verlag GmbH, 2007); http://dx.doi.org/10.1002/9783527617623.fmatter.
11. Mehrling T., Grebenyuk J., Tsung F.S., et al. *Phys. Rev. Spec. Top. Accel. Beams*, **15**, 111303 (2012).
12. Pousa A.F., de la Ossa A.M., Assmann R.W. *Sci. Rep.*, **9**, 17690 (2019).
13. Thévenet M., Lehe R., Schroeder C.B., et al. *Phys. Rev. Accel. Beams*, **22**, 051302 (2019).
14. Veisman M.E., Kuznetsov S.V., Andreev N.E. *Quantum Electron.*, **46** (4), 288 (2016) [*Kvantovaya Elektron.*, **46** (4), 288 (2016)].
15. Andreev N., Baranov V., Matevosyan H. *Laser Part. Beams*, **35** (4), 569 (2017).
16. Esarey E., Shadwick B.A., Catravas P., Leemans W.P. *Phys. Rev. E*, **65**, 056505 (2002).
17. Pugacheva D.V., Andreev N.E. *Quantum Electron.*, **48** (4), 291 (2018) [*Kvantovaya Elektron.*, **48** (4), 291 (2018)].
18. Andreev N.E., Kuznetsov S.V., Pogorelsky I.V. *Phys. Rev. Spec. Top. Accel. Beams*, **3**, 021301 (2000).
19. Andreev N.E., Baranov V.E., Cros B., et al. *Nucl. Instrum. Methods Phys. Res., Sect. A*, **653** (1), 66 (2011).
20. Kuznetsov S.V. *Plasma Phys. Rep.*, **37**, 218 (2011) [*Fiz. Plazmy*, **37**, 242 (2011)].
21. Kuznetsov S.V. *JETP*, **123**, 169 (2016) [*Zh. Eksp. Teor. Fiz.*, **150**, 195 (2016)].

22. Edvards D., Syphers M. *An Introduction to the Physics of High Energy Accelerators* (Wiley, 1993).
23. Andreev N., Kuznetsov S. *IEEE Trans. Plasma Sci.*, **28**, 1211 (2000).
24. Andreev N., Kuznetsov S. *IEEE Trans. Plasma Sci.*, **36**, 1765 (2008).
25. Andreev N.E., Gorbunov L.M., Kirsanov V.I., et al. *Phys. Plasmas*, **4** (4), 1145 (1997).
26. Zheng-Hong L. *Chin. Phys. B*, **17** (3), 901 (2008).
27. Sprangle P., Esarey E. *Phys. Fluids B*, **4** (7), 2241 (1992).
28. Andreev N.E., Kirsanov V.I., Gorbunov L.M. *Phys. Plasmas*, **2** (6), 2573 (1995).
29. Esarey E., Schroeder C.B., Shadwick B.A., et al. *Phys. Rev. Lett.*, **84**, 3081 (2000).
30. Andreev N.E., Kuznetsov S.V. *Plasma Phys. Controlled Fusion*, **45** (12A), A39 (2003).
31. Andreev N.E., Chegotov M.V., Veisman M.E. *IEEE Trans. Plasma Sci.*, **28** (4), 1193 (2000).