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Structure of a chaotic tangle of quantum vortices in turbulent superfluid liquids and in a Bose–Einstein condensate

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Abstract. **Based on the theory of the thermodynamic equilibrium in a system of quantum vortices in superfluid liquids and in a Bose–Einstein condensate in the presence of a counterflow of normal and superfluid components, we investigate the structure of a chaotic tangle of quantum vortices in turbulent superfluid liquids. Using the characteristic functional method, the properties of hydrodynamic vortex filaments are examined. It is shown that the average curvature of the vortex lines is on the order of the intervortex distance, with the proportionality coefficient being independent of the counterflow velocity. It is found that the degree of anisotropy of the vortex loops does not depend on the applied counterflow velocity. The obtained results explain the origin of the anisotropy as well as the relationship between the curvature of the lines and the intervortex space and their dependence on the parameters of the problem.**

Keywords: *Bose–Einstein condensate, quantum vortices, superfluid turbulence, topological defects.*

1. Introduction

Interest in thermodynamically equilibrium quantum vortices is explained by several reasons. First of all, thermodynamic (produced by thermal fluctuations) quantum vortices determine many physical properties of quantum liquids, such as phase transition or kinetic properties (see, for example, [1]). In this sense, the problem under study is of undoubted interest.

Another motivation is associated with the theory of quantum turbulence, namely, the problem of stochastic dynamics of quantum vortex filaments in flowing (and counterflowing) superfluid liquids and in a Bose–Einstein condensate. The theory of quantum turbulence, initiated by the works of Feynman [2] and Onsager [3], has always attracted rapt attention of physicists. Important stages in the study of quantum turbulence are the invention of a macroscopic theory (Vinen [4]), as well as the first numerical work (Schwartz [5]), in which various characteristics of a vortex tangle were obtained. To date, the theory of quantum turbulence is an actively developing discipline with a large number of applications in various fields of physics, from research on ultracold atoms and heavy ions to classical turbulence and physics of neutron stars. An example is the theory of classical turbulence [6], the theory of cosmic strings [7], the theory of dislocations in solids [8], and the theory of phase transitions [1]. The concept of quantum turbulence is also used in studies of quark–gluon plasma [9] and neutron stars [10]. It is worth mentioning two recent international conferences that discussed the problems listed above: INT Program INT-19-1a 'Quantum Turbulence: Cold Atoms, Heavy Ions, and Neutron Stars' (Seattle, USA, 2019), http://www.int.washington.edu/PROGRAMS/19-1a/ and 'Turbulence of All Kinds' (Osaka City University, 2020), https://sites.google.com/view/toak2.

Quantum turbulence in a Bose–Einstein condensate is usually examined by using a macroscopic wave function obeying the nonlinear Schrödinger equation. Studies of the dynamics of ultracold atoms, both theoretical and experimental, are very numerous (see, for example, [11–14].

The bulk of our ideas about the structure and dynamics of vortex tangles is obtained from experiments and direct numerical simulations. Theoretical research is far behind. Of course, this situation is unsatisfactory for theorists, but it is equally unsatisfactory in general. Indeed, for numerical studies and experimental measurements to be more efficient and reliable, scientists obviously need support and new ideas based on analytical investigations. The absence of a consistent theory is explained, firstly, by the unusual complexity of the problem and, secondly, by the fact that the dynamics of vortices is described by a phenomenological approach, and many elements of evolution, for example, reconnection of filaments, are artificial (see review [15]). Thus, there is a need for some particular approach to the general problem. An important version of this approach is the study of thermodynamic equilibrium in a system of quantum vortices in the case of a counterflow of normal and superfluid components. As was shown earlier, this problem has an analytical solution (see [16] and references therein); therefore, there are reasons to clarify many aspects of structure and dynamics.

In this work, we study the problem of the structure of a thermodynamically equilibrium chaotic tangle of quantum vortex filaments in superfluid liquids and in a Bose–Einstein condensate in the presence of a counterflow of normal and superfluid components. Based on the Gibbs distribution determined earlier, we employ the characteristic functional method to obtain the results on the structure of the vortex tangle. In particular, the average curvature of the vortex lines, as well as the anisotropy coefficient of the vortex tangle, is calculated. The results are compared with similar data obtained for the case of quantum turbulence.

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2. Quantum turbulence, vortex tangle properties

The term 'quantum turbulence' was introduced by Feynman in his fundamental work [2] (see also [4]). He described this phenomenon as the appearance of an unordered set of quantum vortex lines or a vortex tangle in superfluid helium (He II), with the tangle resisting the flow of a normal component that transfers entropy. Vortex filaments are one-dimensional structures around which a superfluid component having a velocity $\boldsymbol{v}_{\rm s}$ can circularly move with quantized circulation $\int \boldsymbol{v}_{\rm s} {\rm d}\boldsymbol{l} = n\kappa$. A vortex filament can be described in a parametric form by the function $\boldsymbol{s}(\xi, t)$, where $\boldsymbol{s}$ are the radius vectors of the points of the line, and the parameter ξ 'recalculates' the points of the line; often, the quantity ξ is a parameter of the arc length. The set of lines $\{\boldsymbol{s}(\xi_j, t)\}$ (j is the loop number) evolves, obeying the equations of motion and boundary conditions. The subscript j denotes the number of the vortex loops. Sometimes, for brevity, we will denote the vortex configuration as $\boldsymbol{s}(\xi)$, implying the combination of all the loops, $\boldsymbol{s}(\xi) = \cup_j \boldsymbol{s}_j(\xi_j)$ included in the vortex tangle. To get an idea of the physics of quantum turbulence and the structure of a vortex tangle, we refer readers to our review article [15].

We present several results (numerical and experimental [5, 17]) concerning the structure of a vortex tangle. One of the most widely accepted results is that the density L of the vortex lines (total length of the filaments per unit volume) is proportional to the square of the difference between the velocities of the normal ($\boldsymbol{v}_{\rm n}$) and superfluid ($\boldsymbol{v}_{\rm s}$) components $\boldsymbol{v}_{\rm ns}$ (or counterflow velocity):

$$L = \gamma^2 v_{\rm ns}^2, \tag{1}$$

where $\gamma = \gamma(t)$ is some temperature-dependent function.

Another example concerns the average curvature of the vortex lines, R^{-1}. The value of R has the order of the intervortex (interline) distance $\delta = L^{-1/2}$:

$$1/R^2 = \langle \boldsymbol{s}_j''(\xi_j)\, \boldsymbol{s}_j''(\xi_j) \rangle = c_2^2(T)L, \tag{2}$$

where the primes denote the derivatives with respect to ξ. In the temperature range $1{\rm K} \leqslant T \leqslant 2{\rm K}$ (the usual interval for numerical studies of quantum turbulence), the coefficient $c_2(T)$ varies from 3.5 to unity. Consequently, for low temperatures, at which the interaction of the vortices with the normal component is weak, the vortex lines are more 'broken'. It is noteworthy that $c_2(T)$ is independent of the applied velocity $\boldsymbol{v}_{\rm ns}$.

Then, in numerical simulation and in the experiment, anisotropy of vortex filaments was observed with respect to the vector $\boldsymbol{v}_{\rm ns}$ directed along the z axis:

$$\langle \boldsymbol{s}_x'(\xi_j)\, \boldsymbol{s}_x'(\xi_j) \rangle = I_{xx}, \quad \langle \boldsymbol{s}_y'(\xi_j)\, \boldsymbol{s}_y'(\xi_j) \rangle = I_{yy},$$
$$\langle \boldsymbol{s}_z'(\xi_j)\, \boldsymbol{s}_z'(\xi_j) \rangle = I_{zz}. \tag{3}$$

Despite numerous works on quantum turbulence, there are still no studies in which relations (1)–(3) would be obtained on the basis of any consistent theory. The origin and physical meaning of these relationships are unclear. Quite a mystery is the fact that the parameters I_{xx}, I_{yy}, and I_{zz}, characterising anisotropy are independent of the value of the applied counterflow velocity $\boldsymbol{v}_{\rm ns}$, although the presence of a counterflow is the source of anisotropy.

Continuing sequentially our research, we set the goal to study the problem of the structure of a vortex tangle for thermodynamically equilibrium vortex filaments. In our work [18, 19], we used the Langevin approach to demonstrate the existence of thermodynamic equilibrium of an ensemble of vortex filaments in quiescent superfluid helium, as well as in the presence of a counterflow of normal and superfluid components. We briefly outline the main results. Based on the Langevin approach for the dynamics of vortex filaments, we have shown that the corresponding Fokker–Planck equation has a solution in the form of the Gibbs distribution for the probability distribution functional:

$$P(\{\boldsymbol{s}(\xi)t) = N \exp\left[-\frac{E(\{s\}) - \boldsymbol{P}(\boldsymbol{v}_{\rm n} - \boldsymbol{v}_{\rm s})}{k_{\rm B}T}\right], \tag{4}$$

where N is the normalisation factor. The energy $E(\{\boldsymbol{s}\})$and the Lamb momentum $\boldsymbol{P}(\{\boldsymbol{s}\})$are defined as integrals along the line (see, for example, [20]):

$$E(\boldsymbol{s}) = \frac{\rho_{\rm s}\kappa^2}{8\pi}\int\int \frac{\boldsymbol{s}'(\xi)\boldsymbol{s}'(\xi')}{|\boldsymbol{s}(\xi) - \boldsymbol{s}'(\xi')|}{\rm d}\xi{\rm d}\xi',$$
$$\boldsymbol{P} = \frac{\rho_{\rm s}\kappa}{2}\int \boldsymbol{s}(\xi) \times \boldsymbol{s}'(\xi){\rm d}\xi. \tag{5}$$

Here $\rho_{\rm s}$ is the density of the superfluid component and k is the circulation quantum.

In our previous work [16], we used the Gibbs distribution (4) to construct a statistical sum corresponding to the Gibbs distribution and to calculate the density L of vortex filaments. The obtained result coincides with dependence (1) observed in the experiment. Note that the quantity L is related to the parameters of the problem as follows:

$$L = f(T)\frac{\kappa^2}{\varepsilon_{\rm v}^2}\rho_{\rm s}^2 v_{\rm ns}^2, \tag{6}$$

where $f(T)$ is a temperature function of the order of unity and $\varepsilon_{\rm v}$ is the energy of the unit length of the filament in the local approximation [16].

3. Characteristic functional

To calculate the structural characteristics (2) and (3) of a vortex tangle, we need a powerful analytical tool, often used in statistical problems, the so-called characteristic functional (see, for example, [21]). For a set of chaotic vortex lines, this approach was proposed by Migdal [22, 23]. Following these works, we define the characteristic functional $W(\{\boldsymbol{P}_j(\xi_j)\})$ as the average:

$$W(\{\boldsymbol{P}j(\xi j)\}) = \left\langle \exp\left[{\rm i}\int_0^l \boldsymbol{P}(\xi)\boldsymbol{s}'(\xi){\rm d}\xi\right]\right\rangle.$$

Averaging can be performed using the probability distribution functional (4) through the path integral:

$$W(\{\boldsymbol{P}_j(\xi_j)\}) = \int {\rm D}\boldsymbol{s}(\xi_j)P(\boldsymbol{s}(\xi_j))\exp\left[{\rm i}\int_0^l \boldsymbol{P}(\xi)\boldsymbol{s}'(\xi){\rm d}\xi\right]. \tag{7}$$

The characteristic functional (7) allows us to calculate the average of any value (depending on the configuration of the vortex lines) by simple functional differentiation. For example, the average tangent vector $\langle \boldsymbol{s}_{j\alpha}'(\xi_j) \rangle$ or the correlation function of the tangential vectors of various elements of the vortex filament $\langle s'_{j\alpha}(\xi_j)\, s'_{j\beta}(\xi_j)\rangle$ are easily expressed in terms of the characteristic functional $W(\{\boldsymbol{P}_j(\xi_j)\})$ (7) in accordance with the rules

$$\langle s'_{j\alpha}(\xi_j)\rangle = \frac{\delta W}{\mathrm{i}\delta \boldsymbol{P}_j^{(\alpha)}(\xi_j)}\bigg|_{\boldsymbol{P}=0}, \tag{8}$$

$$\langle s'_{j\alpha}(\xi_j)\, s'_{j\beta}(\xi_j)\rangle = \frac{\delta^2 W}{\mathrm{i}\delta \boldsymbol{P}_j^{(\alpha)}(\xi_{j1})\mathrm{i}\delta \boldsymbol{P}_j^{(\beta)}(\xi_{j2})}\bigg|_{\boldsymbol{P}=0}. \tag{9}$$

While the characteristic functional is determined by the averaging procedure as an auxiliary quantity, it plays an essential independent role in stochastic theories. For example, in the problems of statistical physics of many-particle systems, the use of the characteristic functional (in this theory it is usually called the generating functional) allows us to obtain a brief description of statistical properties in terms of the Green's function and equations for them (see, for example, [21, 22]). Another example is the theory of classical turbulence, where the basic kinetic equation for the characteristic functional (called the Hopf equation) is derived directly from the Navier–Stokes equations without resorting to a distribution function that is in any case unknown (see, for example, [24, 25]).

We proceed to the calculation of the characteristic functional. At this stage, we restrict ourselves to the expression for energy in the local approximation [16], and also consider the case of loops of the same size l. As will be seen from what follows, despite these limitations, we obtain results related to the structural properties of the vortex tangle and close to those observed in the experiment. In the local approximation, the energy of the vortex loop is proportional to the length of the filament, $E_{\mathrm{loc}} = \varepsilon_{\mathrm{v}} \int |\boldsymbol{s}'(\xi)|\,\mathrm{d}\xi = \varepsilon_{\mathrm{v}} l$. The energy of the unit length of the vortex line is expresses as

$$\varepsilon_{\mathrm{v}} = \frac{\rho_{\mathrm{s}}\kappa^2}{4\pi}\ln\left(\frac{\langle R\rangle}{a_0}\right), \tag{10}$$

where a_0 is the radius of the core of the vortex filament, and the upper cutoff parameter for the logarithm $\langle R\rangle$ coincides with the average radius of curvature of the vortex filament, which is related to the filament density L as $\langle R\rangle \approx L^{-1/2}$.

As the next step, we use the so-called Gaussian approximation, which is widely used in the theory of polymer chains. The essence of this approach is to ease the strict condition $|\boldsymbol{s}'(\xi)| = 1$ and change it using a fuzzy (Gaussian) distribution of the link length with the same integral value (see, for example, book [26]). Given that, we will represent the local energy in the form

$$E_{\mathrm{loc}} = \varepsilon_{\mathrm{v}} \int (\boldsymbol{s}'(\xi))^2\,\mathrm{d}\xi.$$

In the Gaussian approximation, the probability distribution functional (4) has the form:

$$P\{\boldsymbol{s}(\xi)\} = N\exp\Big[-\beta\varepsilon_{v}\int(\boldsymbol{s}'(\xi))^2\,\mathrm{d}\xi + \beta\boldsymbol{v}_{\mathrm{ns}}\frac{\rho_{\mathrm{s}}\kappa}{2}\int \boldsymbol{s}(\xi)\times\boldsymbol{s}'(\xi)\,\mathrm{d}\xi\Big]. \tag{11}$$

The probability distribution functional (11) should be supplemented with a factor related to the calculation of vortex configurations by lattice models [16]. This procedure can be performed by using the replacemen $\beta\varepsilon_{\mathrm{v}} \to \beta\varepsilon_{\mathrm{v}} + 3/(2a)$, where a is the step of the (cubic) lattice. Next, we will use the redefined value of ε_{v}.

Thus, the probability distribution functional $P\{\boldsymbol{s}(\xi)\}$ has a Gaussian form [quadratic in the variable $\boldsymbol{s}'(\xi)$], and therefore the characteristic functional (7) can be calculated analytically in a general form. Let us briefly describe this procedure. Since the exponential includes derivatives of different orders, it is convenient to perform the one-dimensional Fourier transform along the line $\boldsymbol{s}(\xi) = \sum_p \boldsymbol{s}(p)\exp(\mathrm{i}p\xi)$. The calculation of the characteristic functional (7) based on the probability distribution functional (11) is performed using the standard 'full-square procedure' (see, for example, [27]). Having completed this procedure, we obtain

$$W(\{\boldsymbol{P}(p)\}) = \exp\Big[-\sum_p P^{(\alpha)}(p)\,N^{(\alpha\beta)}(p)\,P^{(\beta)}(-p)\Big]. \tag{12}$$

The matrix $N^{(\alpha\beta)}(p)$ has the form:

$$N^{(\alpha\beta)}(p) =$$

$$\begin{pmatrix} -lp^2\dfrac{\varepsilon_{\mathrm{v}}}{4\beta p^2\varepsilon_{\mathrm{v}}^2+\beta\kappa^2\rho_{\mathrm{s}}^2 v_{\mathrm{ns}}^2} & -lp\kappa\dfrac{\rho_{\mathrm{s}}}{8\beta p^2\varepsilon_{\mathrm{v}}^2+2\beta\kappa^2\rho_{\mathrm{s}}^2 v_{\mathrm{ns}}^2}v_{\mathrm{ns}} & 0 \\ lp\kappa\dfrac{\rho_{\mathrm{s}}}{8\beta p^2\varepsilon_{\mathrm{v}}^2+2\beta\kappa^2\rho_{\mathrm{s}}^2 v_{\mathrm{ns}}^2}v_{\mathrm{ns}} & -lp^2\dfrac{\varepsilon_{\mathrm{v}}}{4\beta p^2\varepsilon_{\mathrm{v}}^2+\beta\kappa^2\rho_{\mathrm{s}}^2 v_{\mathrm{ns}}^2} & 0 \\ 0 & 0 & -\dfrac{1}{4}\dfrac{l}{\beta\varepsilon_{\mathrm{v}}} \end{pmatrix}. \tag{13}$$

The characteristic functional $W(\{\boldsymbol{P}(p)\})$ (12) with the matrix $N^{(\alpha\beta)}(p)$ (13) is the starting point for studying the statistical properties of a vortex tangle.

4. Some statistical properties of a vortex tangle

This section describes some statistical properties of a vortex tangle, which follow from the above-developed formalism. We restrict ourselves to calculating the curvature and anisotropy of the vortex loops. To this end, we perform the inverse Fourier transform in the expression for the matrix $N^{(\alpha\beta)}(p)$ and calculate the correlation function $\langle \boldsymbol{s}'(\xi_1)\boldsymbol{s}'(\xi_2)\rangle$ of orientations of various line elements.

In accordance with the rules of working with the characteristic functional [see formulae (8) and (9)], we can obtain the relations:

$$\langle s'_x(\xi_1)\, s'_x(\xi_2)\rangle = \langle s'_y(\xi_1)\, s'_y(\xi_2)\rangle = \frac{1}{\beta\varepsilon_{\mathrm{v}}}\delta(\xi_1-\xi_2) + \frac{1}{2\beta\varepsilon_{\mathrm{v}}}\frac{\kappa^2\rho_{\mathrm{s}}}{\varepsilon_{\mathrm{v}}}\frac{v_{\mathrm{ns}}}{\kappa}\exp\left(-|\xi_1-\xi_2|\sqrt{\frac{1}{4}\frac{\kappa^2}{\varepsilon_{\mathrm{v}}^2}\rho_{\mathrm{s}}^2 v_{\mathrm{ns}}^2}\right), \tag{14}$$

$$\langle s'_z(\xi_1)\, s'_z(\xi_2)\rangle = \frac{1}{\beta\varepsilon_{\mathrm{v}}}\delta(\xi_1-\xi_2) \tag{15}$$

for the correlation functions of the transverse components of the tangential vectors $\langle s'_x(\xi_1)s'_x(\xi_2)\rangle$ and $\langle s'_y(\xi_1)s'_y(\xi_2)\rangle$, as well as the correlation functions of the longitudinal components $\langle s'_z(\xi_1)s'_z(\xi_2)\rangle$ taken at different points ξ_1 and ξ_2 along the vortex line. The terms with the delta function appeared due to the fact that, in the absence of a relative velocity v_{ns}, the probability distribution functional (11) is the Wiener distribution for random walks. In this sense, the result is trivial.

The second term in expression (14) is associated with the action of the counterflow. In contrast to the term with the delta function, it describes a smooth change in the tangential vector $\boldsymbol{s}'(\xi)$ along the line. The characteristic length over which the tangential vector changes is a quantity that is inverse to the factor in the exponent after the argument $|\xi_1 - \xi_2|$. By definition, it is the average radius with a curvature R. However, the expression under the root is a combination that is included in the formula for the density of vortex lines (6). Combining the result of (14) with formulae (1) and (6), we arrive at the relation

$$c_2(T) = \frac{\delta}{R} = \frac{\kappa\rho_s}{2\gamma\varepsilon_v}. \tag{16}$$

Thus, we obtained a remarkable result: the average radius with a curvature R is of the order of the average intervortex distance $\delta = L^{-1/2}$ [cf. formula (2)]. In the temperature range $1\,\mathrm{K} \leqslant T \leqslant 2\,\mathrm{K}$, the coefficient $c_2(T)$ varies from 5 to 1.5 and does not depend on the value of the applied counterflow velocity. In order of magnitude and in a characteristic change, the functions $c_2(T)$, obtained in our work and in numerical studies on quantum turbulence, correspond to each other.

We discuss the anisotropy of the vortex loops and their orientation with respect to the applied velocity. From the form of the matrix $N^{(\alpha\beta)}(p)$ (13) and the form of the matrix elements for the correlation functions of the tangential vectors in the ξ-space (14) and (15), it follows that the vortex tabgle must be anisotropic, i.e., have different transverse and longitudinal (with respect to the velocity $\boldsymbol{v}_{\mathrm{ns}}$) sizes. However, it is impossible to directly compare the corresponding quantities with the coefficients I_{xx}, I_{yy}, and I_{zz} obtained in numerical calculations [see relation (3)] for the following reason: the correlators $\langle \boldsymbol{s}_x'(\xi_1)\boldsymbol{s}_x'(\xi_2)\rangle$ and $\langle \boldsymbol{s}_z'(\xi_1)\boldsymbol{s}_z'(\xi_2)\rangle$ contain delta functions of the difference of arguments $\xi_1 - \xi_2$, and there arises infinity for coinciding arguments. This is a direct consequence of the random nature of the walk for the vortex line resulting from the Wiener distribution for the probability distribution functional (11).

To overcome this difficulty, we study the three-dimensional configuration and dimensions of the vortex loops. In three-dimensional space, the square of the distance D^2 between the points of the vortex loop separated by the distance along the line can be obtained from the expression for the correlation function of the tangential vector using the rule [27]

$$\int_0^s\int_0^s \langle \boldsymbol{s}'(\xi_1)\,\boldsymbol{s}'(\xi_2)\rangle \mathrm{d}\xi_1\mathrm{d}\xi_2 = \left\langle \int_0^s \boldsymbol{s}'(\xi_1)\,\mathrm{d}\xi_1 \int_0^s \boldsymbol{s}'(\xi_2)\,\mathrm{d}\xi_2 \right\rangle = D^2 .$$

On the other hand, knowing the expressions for the transverse, $\langle \boldsymbol{s}_x'(\xi_1)\boldsymbol{s}_x'(\xi_2)\rangle$, and longitudinal, $\langle \boldsymbol{s}_z'(\xi_1)\boldsymbol{s}_z'(\xi_2)\rangle$, correlation functions [see formulae (14) and (15)], it is possible to calculate the same value componentwise. The calculations lead to the following result:

$$D_{x,y}^2 = \frac{s}{\beta\varepsilon_v} + \frac{3}{\beta\varepsilon_v}\{R[-1 + \exp(-s/R)] + s\}, \tag{17}$$

$$D_z^2 = \frac{1}{\beta\varepsilon_v}s. \tag{18}$$

The contribution to the three-dimensional size of the terms in the correlators associated with the delta function has the form of a random walk, $D \propto \sqrt{s}$, as it should be due to the Wiener nature of the probability distribution functional (11). As for the contribution of the second term related to the action of the applied velocity, for sufficiently long loops ($s \gg R$), the dependence on the radius of curvature and, therefore, on the relative velocity [see relation (16)] disappears. Since our calculations were performed for loops of the same size, it is not entirely clear how the second term in (17) should be modified in the case of loops of different sizes. To do this, one needs to know the size distribution of the loops, but this is a separate problem to be still solved. It is obvious, however, that appropriate manipulations should lead to some temperature dependence. Summing up, we can argue that the vortex tangle really exhibits anisotropic properties and the ratio of its longitudinal and transverse sizes does not depend on the applied velocity, as is the case in quantum turbulence. The physical meaning of this unusual phenomenon is that, depending on the applied velocity, vortex tangles of different intensities, but with the same statistical properties, are generated. In particular, at a very low velocity ($v_{\mathrm{ns}} \to 0$) a small number of vortex loops will be generated, but the degree of their anisotropy will be the same as in a dense vortex tangle.

5. Conclusions

Based on the characteristic functional method, we have studied the properties of an ensemble of quantum vortices in superfluid liquids in the presence of a counterflow of normal and superfluid components. As in our previous work [16], it was established that there are two populations of vortex filaments. These are the thermodynamic vortices generated by thermal fluctuations, and the hydrodynamic vortices associated with the counterflow velocity. Based on the exact calculation of the characteristic functional, we have obtained the correlation functions of tangential vectors that determine the properties of a vortex tangle. In particular, we have shown that the average radius of curvature R of the vortex lines is on the order of the intervortex distance δ. The proportionality coefficient $c_2(T)$ is a function of temperature (on the order of unity) and does not depend on the applied counterflow velocity.

Our calculations of quantitative characteristics show that the degree of anisotropy of the vortex tangle is also independent of the applied counterflow velocity. Previously, similar results were found only numerically for the case of quantum turbulence. So far, no theoretical methods have been developed to obtain such a dependence, and the physical nature of the revealed properties (in particular, independence from the applied velocity) was unclear. The results of this work are in order of magnitude consistent with data for turbulent flows. However, our results have been obtained for the thermodynamically equilibrium case, and it is not yet clear how it relates to the case of quantum turbulence. This issue, as well as other issues concerning the connection of thermodynamic equilibrium with a turbulent flow, are of great interest; it is assumed that they will all be investigated in the future.

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