

INTEGRATED OPTICS

https://doi.org/10.1070/QEL17328

Localised electromagnetic waves in a rhombic waveguide array with competing nonlinearities

A.I. Maimistov

Abstract. **We consider a model of a discrete photonic system representing a quasi-one-dimensional rhombic array of waveguides, where, in addition to the positive cubic nonlinearity, the negative quintic nonlinearity is taken into account for the material of which the central chain of the waveguides is made. The other two waveguide chains are made of an optically linear material. A continual approximation is used to obtain a solution for a system of coupled waves, which describes a wave localised in the transverse direction. In a certain special case, the competition of nonlinearities leads to the formation of a step-shaped distribution of the field intensities over the waveguides.**

Keywords: *localised electromagnetic waves, quasi-one-dimensional rhombic waveguide array, competition of nonlinearities.*

1. Introduction

One of the first examples of a discrete photonic system was considered by Somekh et al. [1]. This is a coupler of channel waveguides connected to each other due to frustrated total internal reflection. It was shown that continuous radiation introduced into the central channel is gradually redistributed over all adjacent channels. As a result, the distribution of intensities over the channels is formed, which resembles the distribution of the intensities of the beams during Raman–Nath diffraction. Various features of light propagation in such discrete photonic systems were later studied theoretically and experimentally in work [2–4]. In Refs [5, 6], instead of waveguides, microcavities were considered. Bloch oscillations of intensities in an array of coupled waveguides were studied in [7–12]. Waveguides in a zigzag array are coupled with the nearest and next neighbours. Wave propagation in such a waveguide array is considered in [13, 14]. A large number of works are devoted to the propagation of waves in an array of nonlinear waveguides [15] and to the formation of discrete solitons in them [16–19]. In addition to the cubic (Kerr) nonlinearity of the materials of which the waveguides were made, other nonlinearities were also examined. For example, quadratically nonlinear media [20, 21] and media with nonlinearities exceeding cubic ones [22–25] were considered. It can be assumed that one-dimensional and two-dimensional waveguide arrays have become popular objects of discrete photonics [26–28].

A new step in the development of discrete photonics is associated with the study of waveguide arrays, the unit cells of which contain more than two sites (waveguides or microcavities, i.e. analogues of photonic points) [29–31]. A simple example is a quasi-one-dimensional rhombic array (Fig. 1). The spectrum of linear waves in this array has three branches, one of which has zero curvature and is called a flat band for this reason. If the light is introduced into the array so that the field in the central chain of the waveguides is zero, and at the sites of the other two surrounding waveguide chains the fields are out-of-phase, then diffraction of the light along the array is absent. The mode corresponding to the indicated field distribution over the waveguides belongs to the flat band. An experimental demonstration of the localisation of the light (absence of diffraction) was carried out using the example of a rhombic array [32–34] and in the case of a two-dimensional Kagome array [35].

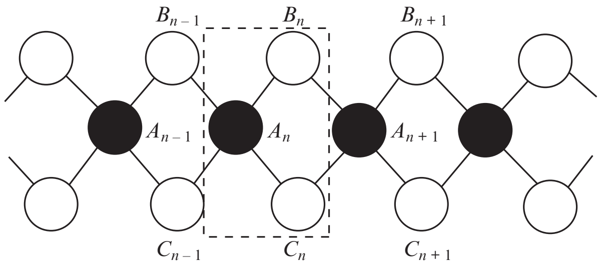

Figure 1. Schematic representation of a binary quasi-one-dimensional rhombic waveguide array. The unit cell is shown by a rectangle.

Based on the theory of coupled waves, Maimistov and Patrikeev [36] showed that in a linear rhombic array, any slight deviation from the field configuration corresponding to the flat-band mode (illumination of at least one site of the central chain of the array or a non-zero sum of the phases of the fields in the sites surrounding the central chain), restores diffraction. In the case of a rhombic array of waveguides with the cubic nonlinearity, it was shown that the modes of the flat band are modulation unstable [37]. Numerical studies of localisation in one unit cell of single and double diamond chains [38] showed that localisation persists for some time, but then the light begins to penetrate into neighbouring waveguides and diffraction is restored. Modulation instability in a two-dimensional rhombic array was studied in [39]. A study of a two-dimensional nonlinear Lieb array, in which the spectrum of lin-

A.I. Maimistov National Research Nuclear University MEPhI, Kashirskoe sh. 31, 115409 Moscow, Russia;
e-mail: aimaimistov@gmail.com

Received 16 March 2020
Kvantovaya Elektronika **50** (9) 844–849 (2020)
Translated by I.A. Ulitkin

ear waves has a flat band, showed that modulation instability leads to localisation of the light in it [40].

In this paper, we consider the localisation of the light in a nonlinear rhombic array in which the central chain of waveguides is made of a material with a positive third-order nonlinear susceptibility and a negative fifth-order susceptibility. The other two waveguide chains are composed of an optically linear material. In addition, it is assumed that the waveguides of either the central chain or the surrounding chains are made of a negative-index material [41, 42]. In this sense, we can speak of a weakly nonlinear binary rhombic array. This model is quite simple for analysis, takes into account the saturation of nonlinearity, the coupling of forward and backward waves, and allows one to take into account the effect of nonlinearity on the modes of the flat band that exists in the rhombic array. In Section 2, equations are formulated that describe the distributions of slowly varying amplitudes of the electric field of an electromagnetic wave over waveguides forming a rhombic array. The system of equations for coupled waves is reduced to one differential-difference equation. Then, considering the waves to be quasi-harmonic, a second-order differential nonlinear equation was obtained in the continuum approximation for the envelope of the distribution of the field amplitudes in the waveguides. In the general case, its solutions represent a wave localised in the array. However, in one exceptional case, a solution was obtained that describes a domain wall (kink) for the intensity distribution over the waveguides of the central chain and localised intensity distributions over the waveguides of the other two chains of the rhombic array. It should be noted that this kind of localisation of an electromagnetic wave is impossible if all waveguides are characterised by a refractive index of the same sign.

2. Basic equations of the model

The propagation of electromagnetic radiation in a waveguide array is usually described based on the theory of coupled waves [43]. An electromagnetic wave is represented by a linear superposition of quasi-harmonic waves localised in the nth waveguide. Using the approximation of slowly varying amplitudes from the wave equation, a differential-difference equation for the amplitudes of the coupled waves A_n in the nth waveguide is derived [1–4, 42]. In the case of a rhombic array (see Fig. 1), the unit cell contains three sites. Consequently, in the approximation of slowly varying amplitudes, we obtain a system of three differential-difference equations for the complex amplitudes A_n, B_n, and C_n of electric fields in the waveguides included in the nth unit cell. Provided that the distances between the nearest waveguides are the same, all coupling constants (or coupling lengths) can be considered equal to each other. Let the waveguide lengths be small, so that the difference in group velocities can be neglected by setting them equal to some average velocity v_g. Then the system of equations for slowly varying amplitudes A_n, B_n, and C_n will be written in the form [30, 44]:

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)A_n+\mathrm{e}^{\mathrm{i}\delta_b\zeta}(B_n+B_{n-1})+\mathrm{e}^{\mathrm{i}\delta_c\zeta}(C_n+C_{n-1})=0,$$

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)B_n+\mathrm{e}^{-\mathrm{i}\delta_b\zeta}(A_{n+1}+A_n)=0, \quad (1)$$

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)C_n+\mathrm{e}^{-\mathrm{i}\delta_c\zeta}(A_{n+1}+A_n)=0,$$

where ζ is the spatial coordinate measured in units of the coupling length L_c; and τ is the time measured in units of $t_c = L_c/v_g$. Parameters $\delta_b = (\beta_b - \beta_a)L_c$ and $\delta_c = (\beta_c - \beta_a)L_c$ are the differences in the propagation constants β_b, β_c, and β_a of waves localised in adjacent waveguides B, C, and A, respectively, and are a measure of the discrepancy between the phase velocities of these waves. It is usually assumed that the phase matching condition $\delta_a = \delta_c = 0$ is met. The system of equations (1) was used to analyse the experimental data [32–34]. Its exact solution was obtained in [36].

The generalisation of the model of a rhombic waveguide array consists in taking into account the nonlinear properties of the material of which the waveguides are made. Additionally, it can be assumed that different types of waveguides are characterised by different signs of the refractive index [41]. A simple generalisation of the system of equations (1), which will be considered below, is based on the following system of equations:

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\sigma_1\frac{\partial}{\partial\zeta}\right)A_n+(B_n+B_{n-1})+(C_n+C_{n-1})+G(|A_n|^2)A_n=0,$$

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\sigma_2\frac{\partial}{\partial\zeta}\right)B_n+(A_{n+1}+A_n)=0, \quad (2)$$

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\sigma_2\frac{\partial}{\partial\zeta}\right)C_n+(A_{n+1}+A_n)=0.$$

Here σ_1 and σ_2 are the signs of the refractive index of the material of the waveguides of the corresponding chain. The function$G(|A_n|^2)$ describes the nonlinearity of the material of the waveguides A. For example, for a Kerr medium $G(|A_n|^2) = \mu|A_n|^2$, and the nonlinearity saturation is taken into account by a function of the form [19, 22, 24]

$$G(|A_n|^2)=\frac{\mu|A_n|^2}{1+\rho|A_n|^2}.$$

If the field strength is not very high, then this expression can be written as

$$G(|A_n|^2)=\mu|A_n|^2-\mu\rho|A_n|^4. \quad (3)$$

This type of nonlinearity at $\rho > 0$ was used as an example of competing cubic–quintic nonlinearities to describe solitons in optical fibres [45–49], vortex solitons [50, 51], as well as propagation of extremely short pulses [52].

For the same mean arrays of waveguides B and C, new variables can be determined, i.e. $F_n = B_n + C_n$ and $S_n = B_n - C_n$. Then the equations of the model will be written in the form:

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\sigma_1\frac{\partial}{\partial\zeta}\right)A_n+F_n+F_{n-1}+G(|A_n|^2)A_n=0,$$

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\sigma_2\frac{\partial}{\partial\zeta}\right)F_n+2(A_{n+1}+A_n)=0.$$

The equation for S_n is trivial:

$$\mathrm{i}\left(\frac{\partial}{\partial\tau}+\sigma_2\frac{\partial}{\partial\zeta}\right)S_n=0.$$

Assuming that the radiation coupled into the waveguides is continuous, the system of equations for coupled waves can be represented in the form:

$$\mathrm{i}\sigma_1\frac{\partial}{\partial\zeta}A_n + F_n + F_{n-1} + G(|A_n|^2)A_n = 0,$$
$$\mathrm{i}\sigma_2\frac{\partial}{\partial\zeta}F_n + 2(A_{n+1} + A_n) = 0. \tag{4}$$

Eliminating F_n, the first-order system of equations can be rewritten as a second-order differential-difference equation:

$$\sigma\frac{\partial^2 A_n}{\partial\zeta^2} + 2(A_{n+1} + 2A_n + A_{n-1}) - \mathrm{i}\sigma_2\frac{\partial}{\partial\zeta}\left[G(|A_n|^2)A_n\right] = 0, \tag{5}$$

where $\sigma = \sigma_1\sigma_2$.

If, according to the formula $A_n = (-1)^n\tilde{A}_n$, new fields $\tilde{A}_n$ are introduced, then they will be determined by solutions of the differential-difference equation

$$\sigma\frac{\partial^2 \tilde{A}_n}{\partial\zeta^2} - 2(\tilde{A}_{n+1} - 2\tilde{A}_n + \tilde{A}_{n-1}) - \mathrm{i}\sigma_2\frac{\partial}{\partial\zeta}\left[G(|\tilde{A}_n|^2)\tilde{A}_n\right] = 0. \tag{6}$$

3. Quasi-harmonic waves and the continual approximation

We now use the assumption that the waves in each waveguide satisfy the condition $\tilde{A}_n(\zeta) = \exp(\mathrm{i}\beta\zeta)\mathcal{A}_n$, where β is the propagation constant and $\mathcal{A}_n$ does not depend on the variable ζ. In this case, the equation of coupled waves (6) is reduced to the difference equation

$$\sigma\beta^2\mathcal{A}_n + 2(\mathcal{A}_{n+1} - 2\mathcal{A}_n + \mathcal{A}_{n-1}) - \sigma_2\beta\left[G(|\mathcal{A}_n|^2)\mathcal{A}_n\right] = 0.$$

In the continual approximation, the difference derivatives are replaced by continuous ones according to the rule

$$\mathcal{A}_{n\pm1} = \mathcal{A}(x) \pm l\frac{\partial\mathcal{A}}{\partial x} + \frac{l^2}{2}\frac{\partial^2\mathcal{A}}{\partial x^2} \pm \frac{l^3}{3!}\frac{\partial^3\mathcal{A}}{\partial x^3} + \frac{l^4}{4!}\frac{\partial^4\mathcal{A}}{\partial x^4} \pm \ldots,$$

where $x = nl$; and l is the array spacing. The difference equation obtained above is written as

$$\sigma\beta^2\mathcal{A} - \sigma_2\beta\left[G(|\mathcal{A}|^2)\mathcal{A}\right] + 2l^2\frac{\partial^2\mathcal{A}}{\partial x^2} + \frac{2l^4}{4!}\frac{\partial^4\mathcal{A}}{\partial x^4} + \ldots.$$

If we restrict ourselves to the second derivative with respect to the 'transverse' variable x, then in the continual approximation we obtain the equation

$$2l^2\frac{\partial^2\mathcal{A}}{\partial x^2} + \sigma\beta^2\mathcal{A} - \sigma_2\beta G(|\mathcal{A}|^2)\mathcal{A} = 0, \tag{7}$$

which takes into account the 'lateral dispersion' to a minimum degree.

If all the terms in equation (7) are multiplied by the derivative with respect to x of $\mathcal{A}$ and integrate the result obtained with respect to x, then we will obtain the equation

$$l^2\left(\frac{\partial\mathcal{A}}{\partial x}\right)^2 + \frac{\sigma\beta^2}{2}\mathcal{A}^2 - \frac{\sigma_2\beta}{2}R(|\mathcal{A}|^2) = I_0, \tag{8}$$

where I_0 is the integration constant. Here we used the relation $G(z) = \mathrm{d}R(z)/\mathrm{d}z$, which defines $R(|\mathcal{A}|^2)$ in (8).

Next, we will consider the case when the material of which the type A waveguides are made has nonlinear properties described by expression (3). Then,

$$R(|\mathcal{A}|^2) = \frac{\mu}{2}\left(|\mathcal{A}|^4 - \frac{2}{3}\rho|\mathcal{A}|^6\right).$$

Substitution of this expression in (8) gives the equation

$$\left(\frac{\partial\mathcal{A}}{\partial x}\right)^2 + \frac{\sigma\beta^2}{2l^2}\mathcal{A}^2 - \frac{\sigma_2\beta\mu}{4l^2}\mathcal{A}^4 + \frac{\sigma_2\beta\mu\rho}{6l^2}\mathcal{A}^6 = I_1, \tag{9}$$

where I_1 is the renormalised integration constant. Introducing new variables,

$$y = \sqrt{\frac{\beta^2}{2l^2}}x,\quad \mathcal{A} = A_0 a,\quad A_0^2 = \frac{2\beta}{|\mu|},$$

allows Eqn (9) to be transformed into the equation:

$$\left(\frac{\partial a}{\partial y}\right)^2 + \sigma a^2 - \sigma_2\nu(a^4 - \kappa a^6) = I_3 = \mathrm{const}, \tag{10}$$

where $\nu = \mathrm{sgn}\mu$; and $\kappa = 4\beta\rho/(3|\mu|)$ is a measure of competition between nonlinearities.

Let all waveguides of a rhombic array be made of an optically linear material. In this case, the linearised equation (5) makes it possible to determine the dispersion relation for linear waves $\sigma\beta^2(q) = 8\cos^2(ql/2)$. Hence it follows that the points $q = \pm\pi/l$ are the boundaries of the Brillouin zone. The linearised equation (6) leads to the dispersion relation $\sigma\beta^2(q) = -8\sin^2(ql/2)$. This means that $\sigma = \sigma_1\sigma_2$ must be equal to -1, and equation (7) describes the evolution of a wave packet, the components of which have wave vectors near the boundaries of the Brillouin zone.

4. Localised waves

Localised electromagnetic waves in the considered waveguide array correspond to solutions of Eqn (10) with boundary conditions at $x \to -\infty$:

$$\mathcal{A}(x) \to 0,\quad \frac{\partial\mathcal{A}}{\partial x} \to 0. \tag{11}$$

Consequently, the constant I_3 in this equation is equal to zero, and Eqn (10) takes the form

$$\left(\frac{\partial a}{\partial y}\right)^2 = -\sigma a^2 + \vartheta(a^4 - \kappa a^6), \tag{12}$$

where $\vartheta = \sigma_2\nu$. Hence it follows that for small values of a, the positive left-hand side of this equation is proportional to $-\sigma a^2$. This means that the solution of Eqn (10) for the chosen boundary conditions at $x \to -\infty$ does not exist for $\sigma = 1$. Thus, it is necessary to assume that $\sigma = \sigma_1\sigma_2 = -1$.

The method for solving equation (10) is standard. A substitution is made, i.e. $a = u^{-1/2}$, which leads to the equation

$$\left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 = u^2 + \vartheta(u - \kappa) = \left(u + \frac{1}{2}\vartheta\right)^2 - \frac{1}{4}(1 + 4\vartheta\kappa). \tag{13}$$

Introducing a new variable w such that

$$\left(u + \frac{1}{2}\vartheta\right) = \Delta w,\quad \Delta^2 = \frac{1}{4}(1 + 4\vartheta\kappa),$$

leads to the equation

$$\left(\frac{1}{2}\frac{\partial w}{\partial y}\right)^2 = w^2 - 1,$$

whose solution is known: $w(y) = \cosh[2(y - y_0)]$, where y_0 is the integration constant, which has the meaning of the position of the maximum value of the function $a(y)$. The origin of coordinates on the axis of the array under consideration can be chosen such that $y_0 = 0$. Therefore, $u(y) = \Delta\cosh(2y) - \vartheta/2$. Hence we have

$$a^2(y) = 2[2\Delta\cosh(2y) - \vartheta]^{-1}.$$

Finally, the solution of equation (9) with the chosen boundary condition is written in the form:

$$\mathcal{A}^2(x) = \frac{2A_0^2}{2\Delta\cosh(Kx) - \vartheta}, \tag{14}$$

where

$$A_0^2 = \frac{2\beta}{|\mu|}; \quad K = 2\sqrt{\frac{\beta^2}{2l^2}}; \quad \vartheta = \pm 1.$$

It should be borne in mind that $\mathcal{A}^2(x)$ is the envelope of the radiation intensity distribution over the sites of the waveguide chain A. This distribution is parameterised by the value $\vartheta = \pm 1$. The value $\vartheta = +1$ corresponds to the focusing (defocusing) Kerr nonlinearity and waveguides A made of a positive (negative) index material. Field $\mathcal{A}^2(x)$ is given by the expression

$$\mathcal{A}^2(x) = \frac{2A_0^2}{(1+4\kappa)^{1/2}\cosh(Kx) - 1}.$$

If the nonlinear waveguides were purely Kerr ones ($\rho = 0$), then this solution would be singular, i.e. unreasonable from the point of view of physical properties.

The value $\vartheta = -1$ corresponds to the defocusing (focusing) Kerr nonlinearity of waveguides A made of negative (positive) index material. Field $\mathcal{A}^2(x)$ in this case is given by the expression

$$\mathcal{A}^2(x) = \frac{2A_0^2}{(1-4\kappa)^{1/2}\cosh(Kx) + 1}.$$

Hence, it is seen that the quintic nonlinearity parameter is limited by the inequality $\kappa < 1/4$.

Figure 2 shows the envelopes of localised waves for $\vartheta = +1$ and -1. Moreover, in both cases $\sigma_2 = 1$ and $\kappa = 0.1$. It can be seen that the maximum values of the amplitude $\mathcal{A}(x)$ for the cases $\vartheta = +1$ and -1 are very different. When the nonlinearity of the medium is purely Kerr one, at $\vartheta = +1$ the amplitude $\mathcal{A}(x)$ becomes infinitely large in one of the waveguides. It would be wrong to be limited here by cubic (Kerr) nonlinearity.

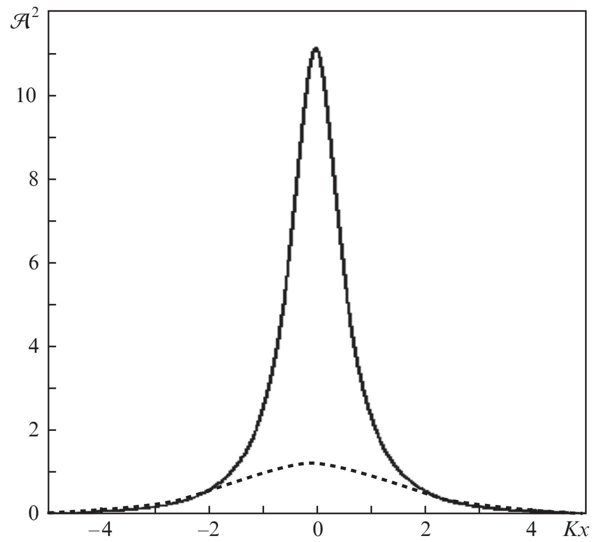

Figure 2. Localised wave envelopes at $\vartheta = +1$ (solid line) and $\vartheta = -1$ (dashed line).

Setting $F_n(\zeta) = (-1)^n\mathcal{F}\exp(i\beta\zeta)$ in the considered approximation and using equation (4), the following expression can be obtained for the envelope $\mathcal{B}(x)$:

$$2\mathcal{B}(x) = \mathcal{F}(x) = -\frac{2l}{\sigma_2\beta}\frac{\partial\mathcal{A}}{\partial x}.$$

The envelopes of the field distributions in the array of waveguides A and B in the transverse and longitudinal directions are shown in Fig. 3.

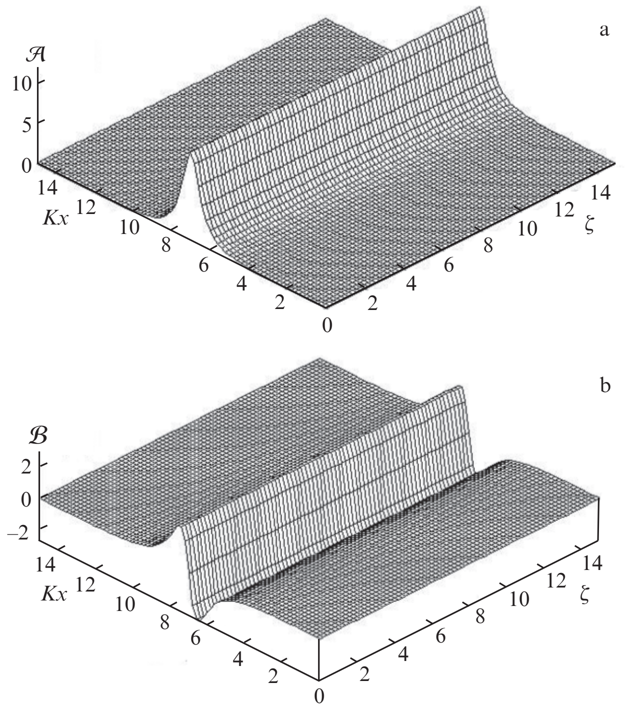

Figure 3. Localised wave envelopes for (a) $\mathcal{A}(x)$ and (b) $\mathcal{B}(x)$ at $\vartheta = +1$.

5. Domain walls

Above, we considered the case when the quintic nonlinearity parameter is limited by the inequality $4\kappa < 1$. The case $4\kappa = 1$ must be considered separately. Here the variable $u(y)$ is defined by the equation

$$\left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 = u^2 - u + \frac{1}{4} = \left(u - \frac{1}{2}\right)^2.$$

Hence,

$$\left(\frac{1}{2}\frac{\partial w}{\partial y}\right)^2 = w^2, \qquad \frac{\partial w}{\partial y} = \pm 2w.$$

The solution to this equation is: $w^{(\pm)} = \exp[\pm 2(y - y_0)]$. Thus, there are two solutions for the amplitude a:

$$a^{(\pm)2}(x) = \frac{2}{1 + 2\exp[\pm 2(y - y_0)]}, \tag{15}$$

where $y - y_0 = K(x - x_0)$. Solution $a^{(+2)}(x)$ describes a wave with an exponentially decreasing envelope at $x \to +\infty$,

$$a^{(+)2}(x) \sim \exp[-2K(x - x_0)]$$

and amplitude tending to a constant value at $x \to -\infty$:

$$a^{(+)2}(x) \sim 2\{1 - 2\exp[-2K(x - x_0)]\}.$$

On the contrary, $a^{(-)2}(x)$ describes a wave with an exponentially decreasing envelope at $x \to -\infty$,

$$a^{(-)2}(x) \sim \exp[-2K(x - x_0)],$$

and an amplitude tending to a constant value at $x \to +\infty$:

$$a^{(-)2}(x) \sim 2\{1 - 2\exp[-2K(x - x_0)]\}.$$

The found solutions can be understood as domain walls (an example is shown in Fig. 4.)

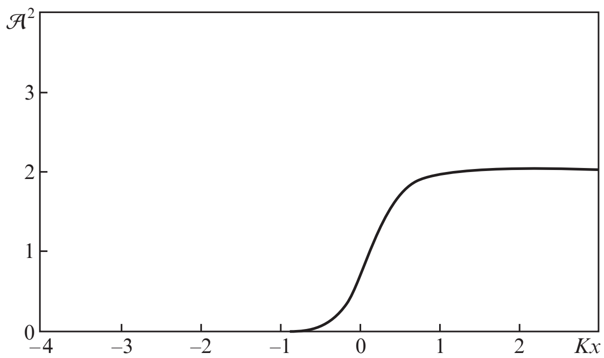

Figure 4. Field profile $\mathcal{A}^{(-)}(x)$ at $4\kappa = 1$.

It should be emphasised that the boundary condition (11) corresponds to the domain wall described by the formula for $a^{(-)}(x)$. The envelope $\mathcal{B}^{(-)}(x)$ looks like a solitary wave with zero asymptotics, localised on the domain wal $\mathcal{A}^{(-)}(x)$.

To consider localised waves in the form of a wave packet of plane waves with wave vectors lying near $q = 0$, one can repeat the derivation of the equation for the envelope based on (5) and use the dispersion equation for linear waves near $q = 0$. For example, in the long-wavelength limit one can obtain the equation

$$2l^2\left(\frac{\partial \mathcal{A}}{\partial x}\right)^2 + (8 - \beta^2)\mathcal{A}^2 + \beta\mu(A^3 - \rho A^5) = 0. \qquad (16)$$

It follows from the dispersion relation for linear waves at the centre of the Brillouin zone that $\max\beta^2 = 8$. Thus, the second term in Eqn (16) is positive, and this equation has no localised solutions with zero asymptotics.

6. Conclusions

Based on the equations of coupled waves for the model of a quasi-one-dimensional rhombic waveguide array, in which, in addition to the positive cubic nonlinearity of the material of which the waveguides are made, the negative quintic nonlinearity is taken into account for the central chain of waveguides, we have obtained a differential equation for the envelope of the field distribution over the waveguides in the continual approximation. This equation accounts for 'lateral dispersion' to a minimum degree. A stationary solution for quasi-harmonic waves is found analytically, which describes a wave localised in the array. In the exceptional case, competing nonlinearities allow the existence of a solution in the form of a kink or domain wall. These waves are wave packets collected from plane waves with wave vectors lying near the boundaries of the Brillouin zone. Due to the periodicity of the waveguide grating, the boundaries $q = -\pi/l$ and $q = +\pi/l$ can be identified.

At the points of the Brillouin zone $q = \pm\pi/l$, the dispersion curves corresponding to the modes of the flat band and the modes of two other ordinary bands merge. Localisation of the light in a waveguide array is due to nonlinear effects, as it happens during self-focusing in continuous media. Deformation of the waveguide array can remove the degeneracy at the points $q = \pm\pi/l$ [27, 53]. It will be necessary to generalise the theory presented here to the case in which the interaction of all modes due to the nonlinearity of the medium is taken into account.

Regarding the solution in the form of a domain wall, it should be noted that it can be obtained experimentally at a very large value of the input radiation amplitude: $A_0^2 \sim \chi^{(3)}/\chi^{(5)}$ in the case of nonresonant nonlinearity or $A_0 \sim A_{\rm sat}$ in the case of the saturation field amplitude [54] for the resonant nonlinearity.

It is essential that the localisation of the electromagnetic wave in the rhombic array turned out to be possible (in the accepted approximation) due to the use of media with positive and negative refractive indices for the manufacture of waveguides. It can be shown that for $\sigma_1 = \sigma_1 = 1$ there is no solution to Eqn (10) with zero asymptotics.

Acknowledgements. The work was supported by the Russian Foundation for Basic Research (Grant No. 18-02-00278).

References

1. Somekh S., Garmire E., Yariv A., et al. *Appl. Phys. Lett.*, **22**, 46 (1973).
2. Pertsch T., Zentgraf T., Peschel U., et al. *Phys. Rev. Lett.*, **88**, 093901 (2002).
3. Staron G., Weinert-Raczka E., Urban P. *Opto-Electron. Rev.*, **13**, 93 (2005).
4. Röpke U., Bartelt H., Unger S. *Appl. Phys. B.*, **104**, 481 (2011).
5. Christodoulides D.N., Efremidis N.K. *Opt. Lett.*, **27** (8), 568 (2002).
6. Armaroli A., Feron P., Dumeige Ya. *Phys. Rev. A*, **98**, 013848 (2018).
7. Peschel U., Pertsch T., Lederer F. *Opt. Lett.*, **23**, 1701 (1998).
8. Szameit A., Pertsch T., Nolte S., et al. *J. Opt. Soc. Am.*, **24**, 2632 (2007).
9. Longhi S. *Opt. Lett.*, **34**, 2174 (2009).
10. Gang Wang. *J. Opt.*, **16**, 015502 (2014).
11. Yulin A., Driben R., Meier T. *Phys. Rev. A*, **96**, 033827 (2017).
12. Wei Feng Zhang, Xiao Zhang, Kartashov Ya.V., Xianfeng Chen, Fangwei Ye. *Phys. Rev. A*, **97**, 063845 (2018).
13. Efremidis N.K., Christodoulides D.N. *Phys. Rev. B*, **65**, 056607 (2002).
14. Kazantseva E.V., Maimistov A.I. *Quantum Electron.*, **43**, 807 (2013) [*Kvantovaya Elektron.*, **43**, 807 (2013)].
15. Lederer F., Stegeman G.I., Christodoulides D.N., Assanto G., Segev M., Silberberg Ya. *Phys. Rep.*, **463**, 1 (2008).
16. Shen Y., Kevrekidis P.G., Srinivasan G., Aceves A.B. *J. Phys. A: Math. Theor.*, **49**, 295205 (2016).
17. Auditore A., Conforti M., De Angelis C., Aceves A.B. *Opt. Commun.*, **297**, 125 (2013).

18. Kivshar Yu.S., Flytzanis N. *Phys. Rev. A*, **46**, 7972 (1992).
19. Cuevas J., Eilbeck J.C. *Phys. Lett. A*, **358**, 15 (2006).
20. Darmanyan S., Kobyakov A., Lederer F. *Phys. Rev. E*, **57**, 2344 (1998).
21. Iwanow R., Schiek R., Stegeman G.I., Pertsch T., et al. *Phys. Rev. Lett*., **93**, 113902 (2004).
22. Herrmann J. *J. Opt. Soc. Am. B*, **8**, 1507 (1991).
23. Afanasjev V.V., Chu P.L., Kivshar Y.S. *Opt. Lett*., **22**, 1388 (1997).
24. Veretenov N.A., Rosanov N.N., Fedorov S.V. *Phys. Rev. Lett.*, **117**, 183901 (2016).
25. Abdullaev F.Kh., Salerno M. *Phys. Rev. E*, **97**, 052208 (2018).
26. Longhi S. *Laser & Photonics Rev*., **3**, 243 (2009).
27. Szameit A. Nolte S. *J. Phys .B: Atom., Molec. Opt. Phys.*, **43**, 163001 (2010).
28. Maimistov A.I. *Nonlin. Phenom. Compl. Syst.*, **19**, 358 (2016).
29. Morales-Inostroza L. Vicencio R.V. *Phys. Rev. A*, **94**, 043831 (2016).
30. Flach S., Leykam D., Bodyfelt J.D., Matthies P., Desyatnikov A.S. *Europhys. Lett.*, **105**, 30001 (2014).
31. Leykam D., Andreanov A., Flach S. *Adv. Phys.: X*, **3**, 1473052 (2018).
32. Mukherjee S., Thomson R.R. *Opt. Lett*., **40** (23), 5443 (2015).
33. Mukherjee S., Spracklen A., Choudhury D., Goldman N., et al. *Phys. Rev. Lett*., **114**, 245504 (2015).
34. Vicencio R.A., Cantillano C., Luis R.A., et al. *Phys. Rev. Lett*., **114**, 245503 (2015).
35. Yuanyuan Zong, Shiqiang Xia, Liqin Tang, et al. *Opt. Express*, **24** (8), 8877 (2016).
36. Maimistov A.I., Patrikeev V.A. *J. Phys.: Conf. Ser.*, **737**, 012008 (2016).
37. Maimistov A.I. *J. Opt*., **19**, 045502 (2017).
38. Zegadlo K., Nir Dror, Hung N.V., Trippenbach M., Malome B.M. *Phys. Rev. E*, **96**, 012204 (2017).
39. Porsezian K., Ali Shafeeque A.K., Maimistov A.I. *J. Opt. Soc. Am. B*, **35**, 2057 (2018).
40. Lazarides N., Tsironis G.P. *Phys. Rev. B*, **96**, 054305 (2017).
41. Sarychev A.K., Shalaev V.M. *Electrodynamics of Metamaterials* (Singapore: World Scientific, 2007).
42. Maimistov A.I., Gabitov I.R. *Eur. Phys. J. Special Topics*, **147**, 265 (2007).
43. Tamir T. (Ed.) *Integrated Optics* (Berlin: Springer-Verlag, 1975; Moscow: Mir, 1978).
44. Longhi St. *Opt. Lett*., **39**, 5892 (2014).
45. Azimov B.S., Sagatov M.M., Sukhorukov A.P. *Sov. J. Quantum Electron*., **21**, 93 (1991) [*Kvantovaya Elektron*., **18**, 104 (1991)].
46. Mihalache D., Mazilu D., Crasovan L.-C., Malomed B.A., Lederer F. *Phys. Rev. E*, **61**, 7142 (2000).
47. Abdullaev F.Kh., Garnier J. *Phys. Rev. E*, **72**, 035603 (R) (2005).
48. Woo-Pyo Hong. *Opt. Commun*., **194**, 217 (2001).
49. Woo-Pyo Hong. *Opt. Commun*., **213**, 173 (2002).
50. Mihalache D., Mazilu D., Crasovan L.-C., Malomed B.A., Lederer F., Torner L. *Phys. Rev. E*, **68**, 046612 (2003).
51. Desyatnikov A., Maimistov A., Malomed B. *Phys. Rev. E*, **61**, 3107 (2000).
52. Maimistov A.I. *Opt. Spectrosc*., **94**, 251 (2003) [*Opt. Spektrosk*., **94**, 281 (2003)].
53. Dreisow F., Szameit A., Heinrich M., Pertsch T., Nolte D., et al. *Phys. Rev. Lett*., **102**, 076802 (2009).
54. Pantell R., Puthoff H. *Fundamentals of Quantum Electronics* (New York: Wiley, 1969; Moscow: Mir, 1972).