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Recursive method for solving the inverse problem of photocount statistics

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Abstract. **An analytical recursive method for solving the inverse problem of photocount statistics, that is, for reconstructing the photon-number distribution from the photocount distribution, is proposed for few-photon light. It is shown that if the inverse problem is represented as a system of linear equations, then for finite distributions one can obtain a recurrence formula relating the photocount distribution and the photon-number distribution without restrictions on the quantum efficiency of photodetection.**

Keywords: *quantum optics, photocount statistics, inverse problem, recursive method.*

1. Introduction

The photon counting is one of the main methods for measuring the characteristics of light fields in quantum optics. This method of recording optical radiation has more than a century of history [1], but it began to develop especially intensively soon after the advent of lasers. To date, the photon counting method is widely used both in applied [2–6] and in fundamental research [7–10]. The method is based on the photoelectric effect, in which an electron is emitted from the cathode surface after the absorption of one or a few photons. The photon counting method counts the number of photoelectrons ‘knocked out’ by the light incident on the photocathode over a certain fixed time interval T. When recording continuous or periodic pulsed radiation, multiple repetitions of measurements are possible. In this case, the value of the photon counting method increases, since it becomes possible to measure a more informative characteristic of the photoelectric effect, namely, the photocount statistics, that is, the probability distribution Q_m that m photoelectrons will be detected during the measurement time T.

Recently, in connection with the development of quantum optical technologies, the significance of information on the statistical properties of few-photon radiation has increased, which has stimulated additional interest in a deeper study of both the direct and inverse problems of photocount statistics. From a practical point of view, the inverse problem seems to be more important, namely, the reconstruction of the photon-number distribution from the measured photocount statistics. The importance of this problem, in particular, is due to the wide use in current optical technologies and in quantum optics [11–13] of few-photon light sources, whose shot noise is comparable to the average intensity of the light signal or even higher. The energy characteristics of such sources can be obtained most easily using photoelectric measurements.

As follows from the quantum theory of the photoelectric effect, knowledge of the photocount statistics is not enough for complete reconstruction of the quantum state of the field, generally determined by a density matrix, but it is sufficient to determine its diagonal elements [14, 15]. In terms of information value, knowing the density matrix diagonal elements for few-photon light is similar to knowing the power for cw radiation or the intensity profile for pulsed radiation from bright light sources. The photocount statistics, in principle, allows obtaining a complete energy description of few-photon light sources, but for this purpose, the inverse problem of photocount statistics must be solved.

Already at the initial stage of the development of the photon counting method for classical light, in addition to the direct problem of finding the photocount statistics from the intensity distribution, of interest was the inverse problem, that is, the reconstruction of the light intensity distribution from the photocount statistics. Apparently, for the first time, this problem was solved in [16]. The authors proceeded from the semi-classical Mandel formula [17], according to which the photocount distribution is the radiation intensity distribution averaged over the Poisson distribution. Later, various approaches to solving this problem were developed (see, e.g., [18–21]). These approaches were also based on the Mandel formula.

In the quantum description, which is valid not only for classical, but also for nonclassical light, the intensity corresponds to the photon number operator, and the intensity distribution turns into a discrete photon-number distribution P_n, that is, the probability that during the measurement time T there will be n photons in the light flux. In this case, the direct problem of the photocount statistics is to find the photocount distribution Q_m from a given photon-number distribution P_n. These distributions, according to [15, 22–24], are related through the Bernoulli transformation:

$$Q_m = \sum_{n \geqslant m} C_n^m \eta^m (1-\eta)^{n-m} P_n, \qquad (1)$$

where η is the quantum efficiency of photodetection; and $C_n^m = n!/[m!(n-m)!]$ is the binomial coefficient.

Thus, in the case of few-photon light, the inverse problem of the photocount statistics should be understood as finding the photon-number distribution P_n in Eqn (1) from a given

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photocount distribution Q_m. Two analytical methods are known for solving this problem. The first of them is based on a direct inversion of the Bernoulli transformation [23], the second on the generating-function approach [25].

In this paper, we would like to draw attention to the possibility of associating the inverse problem of photocount statistics with a system of linear equations and the possibility of solving it by the recursive method, which may be useful for reconstructing finite distributions of photons.

2. Recurrence formula relating the distribution of photons with the photocount distribution

The solution of the direct problem of photocount statistics (1) can be represented in matrix form [15]:

$$Q_m = D_{mn}(\eta) P_n, \tag{2}$$

where the triangular matrix $D_{mn}(\eta)$ has the form

$$D_{mn}(\eta) = \begin{cases} C_n^m \eta^m (1-\eta)^{n-m}, & n \geqslant m, \\ 0, & n < m. \end{cases} \tag{3}$$

Photon-number distributions can be finite, for example, the number distribution of photons emitted by an ensemble of quantum dots [26] or fluorescent molecules [27], and infinite, such as the Poisson and Bose–Einstein distributions, as well as the photon-number distribution in quadrature-squeezed states [28].

For finite distributions, there is such a maximum number of photons N, at which for $n > N$ the value $P_n \equiv 0$. In this case, Eqn (2) can be interpreted as a system of N linear equations. Since the matrix $D_{mn}(\eta)$ is triangular, one can use the well-known Gaussian back substitution and restore the photon-number distribution in a recursive way, starting from the last term.

Based on Eqn (2), we write out expressions for the last few elements of the photocount distribution in explicit form:

$$Q_N = \eta^N P_N,$$

$$Q_{N-1} = \eta^{N-1} P_{N-1} + N\eta^{N-1}(1-\eta) P_N,$$

$$Q_{N-2} = \eta^{N-2} P_{N-2} + (N-1)\eta^{N-2}(1-\eta) P_{N-1} + \frac{N(N-1)}{2}\eta^{N-2}(1-\eta)^2 P_N,$$

and successively express the elements of the photon-number distribution with a smaller number of photons through the elements with a larger number of photons:

$$P_N = \eta^{-N} Q_N,$$

$$P_{N-1} = \eta^{-(N-1)} Q_{N-1} - N(1-\eta) P_N,$$

$$P_{N-2} = \eta^{-(N-2)} Q_{N-2} - (N-1)(1-\eta) P_{N-1} - \frac{N(N-1)}{2}(1-\eta)^2 P_N.$$

Continuing the recursive series, by induction we obtain a recurrence formula for reconstructing the distribution P_n from a given distribution Q_m:

$$P_n = \eta^{-n} Q_n - \sum_{k>n}^{N} C_k^n (1-\eta)^{k-n} P_k. \tag{4}$$

From Eqn (4) it follows that the photon-number distribution can be found by passing successively from the probabilities of a larger number of photocounts to the probabilities of a smaller number of them.

3. Reconstruction of the binomial distribution of photons using the recurrence formula

As an example, consider the finite binomial photon-number distribution emitted by an ensemble of N independent emitters [29]. If we assume that each emitter emits a photon with probability r, then the photon-number distribution will have the form

$$P_n = C_N^n r^n (1-r)^{N-n}. \tag{5}$$

A substitution of this distribution into Eqn (1) easily shows that the photocount distribution is also binomial:

$$Q_m = C_N^m (\eta r)^m (1-\eta r)^{N-m}. \tag{6}$$

By this example, we demonstrate that the obtained recurrence formula allows a correct solution of the inverse problem of the photocount statistics for a finite distribution. According to Eqn (5), we specified the binomial photon-number distribution P_n^{in} with parameters $r = 0.5$ and $N = 10$. Formula (2) was used to calculate the photocount distribution at $\eta = 0.3$, from which the photon-number distribution P_n^{rec} was reconstructed using Eqn (4) and compared with P_n^{in}.

Figure 1 shows the initial photon-number distribution (P_n^{in}) and the one reconstructed using the recurrence formula (4) (P_n^{rec}). The distributions are seen to coincide, which confirms the correctness of Eqn (4) obtained in this work.

4. Conclusions

It is shown that for finite distributions of photons, if a system of linear algebraic equations is associated with the inverse

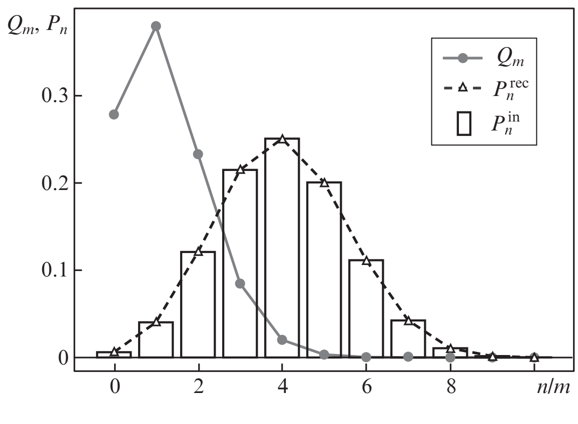

Figure 1. Example of restoring the finite binomial photon-number distribution using the recurrence formula (4). P_n^{in} is the initial photon-number distribution; P_n^{rec} is the reconstructed photon-number distribution; Q_m is the photocount distribution initiated by the light flux with the photon-number distribution P_n^{in}; the calculations were performed at $N = 10$, $\eta = 0.3$, and $r = 0.5$.

problem of photocount statistics, a recurrence formula relating the photocount distribution and the photon-number distribution can be obtained. The formula is numerically tested on a finite binomial photon-number distribution. The photon-number distribution reconstructed using the recurrence formula coincided with the original one up to the 16th decimal place.

The results presented show that, to solve the inverse problem of photocount statistics, along with the inversion of the Bernoulli transformation and the method of generating functions, one can also use the recursive method proposed in this work. Note that caution should be exercised when using the recursive method to estimate the infinite photon-number distributions from the statistics of a finite sample of photocounts. This remark applies not only to the recursive method proposed in this work, but also to all other analytical methods for solving the inverse problem of photocount statistics. The problem of correctly estimating the infinite distributions of photons is related to the stability of the solution to the inverse problem for a finite sample size and, in principle, requires an individual analysis for each specific distribution Q_m.

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