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REVIEW

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Dissipative aspects of extreme nonlinear optics

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Abstract. **Studies of extreme nonlinear optical effects, in which dissipative factors such as absorption and amplification of light in a medium play a fundamental role, are reviewed. The generation of pulses with extremely short duration down to unipolar ones is analysed by tracking the development of ideas related to the self-induced transparency phenomenon, whose practical application becomes real for extremely short pulses in lasers and laser media. Extreme radiation structuring is achieved in dissipative (laser) solitons characterised by a complex topology of phase and polarisation singularities, which is of interest for coding information.**

Keywords: *extremely short radiation pulses, unipolar pulses, self-induced transparency, topological dissipative optical solitons.*

1. Introduction

Although the term ‘nonlinear optics’ itself appeared more than 70 years ago [1], and the first experiments with a substantially nonlinear absorption of saturation were performed even earlier [2], this science acquired its modern form only after the advent of lasers [3]. The subject of its research is the evolution of radiation in a medium or system, the optical properties of which depend on the radiation itself, both the radiation and the medium generally acting as equal partners. Thus, nonlinear optics includes both electrodynamics and physics of media in various aggregate states, which ensures an almost unlimited possibility of the development of this science.

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Over the past 60 years, the extent of research on nonlinear optics has grown tremendously, as can be seen from a comparison of the first [4, 5] and one of the last [6] monographs on this topic. At the same time, transparent media were the main object of nonlinear optics, and absorption was considered as a weak and undesirable effect that impedes the propagation of radiation. Of course, the saturation of absorption by intense radiation was known even in the pre-laser period of optics [2], not to mention its fundamental role in the characteristics of lasing. However, the dissipation-caused phenomena, like transmission waves [7], stood aside from the mainstream of research.

In this review, we would like to emphasise the important stabilising role of dissipative factors in nonlinear optics, which make it possible to realise regimes that are inaccessible to the ‘conservative’ nonlinear optics of transparent media. The issue to be discussed is the stabilisation due to the dynamic balance of such factors as absorption and amplification. Here we will restrict ourselves to only a few, but in our opinion important, areas that reveal the tendencies of modern nonlinear optics. For example, while in the beginning of the laser era in nonlinear optics, mainly the radiation consisting of a small set of quasi-monochromatic quasi-plane waves was considered, in the present review we discuss advancing to extreme regimes concerning both the pulse duration (shortening to unipolar pulses) and the complexity of field topology (topological solitons). In both cases, the principal role belongs to the dissipativity of the medium (system), i.e., the relaxation in it, as well as the energy inflow and outflow. Indeed, even for extremely short pulses, the duration of which is much shorter than the relaxation times of the medium, neglecting dissipativity leads to physically incorrect results. For instance, excitation of medium oscillators by a passage of such a pulse through it would cause an infinitely long emission by these oscillators, i.e., an infinite emitted energy (in the case of solitons, the same behaviour would be also caused by their small perturbations).

Taking into account the already published reviews and monographs, and only briefly mentioning the results presented in them, in the next two sections we will focus on the following issues. First, for extremely short pulses, we present the history and subsequent development of studies of the self-induced transparency effect, up to its manifestation in extremely short and unipolar pulses. Second, we also briefly recall the history of studies of topological optical solitons and more fully present recent results on the topology of vector dissipative solitons (taking into account the polarisation of radiation).

2. Self-induced transparency, prospects for practical application

The main nonlinear optical phenomena discovered soon after the appearance of the first lasers, including harmonic generation, parametric generation, self-focusing, etc., have found practical applications and are widely used now [6]. Different was the fate of self-induced transparency (SIT), demonstrated at the dawn of the laser era by McCall and Hahn [8, 9]. SIT was an extremely unusual phenomenon for those years, partly because it had no direct analogues in nonlinear electronic systems. SIT consists in the fact that a short pulse with a certain intensity, the duration of which is shorter than the relaxation time of the medium polarisation T_2, can propagate in a medium practically without absorption (2π pulse). It is important to note that this phenomenon belongs to the so-called coherent phenomena of the radiation–matter interaction.

At the leading edge, the pulse inverts the medium, and at its trailing edge, because of stimulated emission, the energy absorbed by the medium is returned to the pulse. Therefore, the speed of movement of such a pulse in a medium can be significantly less than the speed of light. One of the most important results of the SIT theory developed in those years is the famous area theorem, which describes the evolution of the pulse envelope area during coherent propagation of a short laser pulse in a resonantly absorbing or amplifying medium [9].

To date, this phenomenon is thoroughly studied theoretically and demonstrated experimentally in various media. Various reviews and monographs [10–13] are devoted to it; the topic is also presented in the Quantum Electronics journal [14, 15].

Among the studies of the SIT phenomenon by Russian authors, we should especially mention the papers by A.I. Maimistov and S.V. Sazonov et al., who continue to study this effect in more depth and under new conditions [16–20].

Unlike many other phenomena of nonlinear optics, the SIT effect has not yet found practical applications and remains a subject of academic research. Although the authors of [21] demonstrated a possibility to compress long laser pulses using SIT, it was easier to obtain short pulses by other methods. The possibility of using SIT for generating short pulses in lasers, which we will discuss below, is not obvious, and the first attempts to implement it were unsuccessful. The lack of visible applications is emphasised, e.g., in Shen's monograph on nonlinear optics [22].

In the first experiments on obtaining short pulses in He–Ne gas lasers, attempts were made to use an additional cell with pure Ne [23]. Absorption at a laser transition in Ne atoms could occur by the SIT mechanism and serve as a source of short pulses as a result of this phenomenon. However, detailed studies have revealed that the emergence of mode locking is due to the incoherent saturation, rather than the coherent SIT phenomenon.

The passive mode locking in a laser is implemented by placing an absorbing medium in its cavity – the so-called saturable absorber. It is a nonlinear passive loss modulator, working based on the saturation of incoherent absorption, which immediately limits the minimum pulse duration to the relaxation times in the absorber. Moreover, saturable absorbers (dyes, semiconductors) possess wide enough absorption lines and exhibit rapid relaxation. From this point of view, consideration of atomic media with narrow absorption lines seems impossible, since they, as it is believed, would not provide a short pulse duration and a tuning of the pulse spectrum within the frequency range of the gain band of the laser active medium. In the literature, including Quantum Electronics, there is a large number of reviews and chapters in monographs devoted to these issues [24–28].

It should be noted that in those years a number of papers were published in which absorbing atomic media were located in the cavities of various types of lasers. In such experiments, the effects of lasing frequency locking were observed, as well as the appearance of a multicomponent spectrum near the absorption lines of various atoms (see, e.g., [29–33]). Different authors interpreted these effects in different ways. The simplest interpretation was associated with inducing refractive index gratings near narrow absorption lines [34, 35]. It was hypothesised that the reason for the change in the lasing spectrum was coherent effects in the interaction of laser radiation with matter [36, 37]. Moreover, in the experiments where a possible coherent mechanism of the influence of atomic transitions on the lasing spectrum was discussed, no effect of self-induced transparency and mode-locking regimes caused by it were found.

At the same time, there was an understanding of the possibility of using SIT to form the mode locking regime. Here we should mention the publication [38] in the Quantum Electronics journal. It theoretically analysed a solid-state laser, in the cavity of which a cell with alkali metal vapour was located. The authors concluded that it is possible to obtain a mode locking regime by forming SIT pulses on resonant transitions of atomic vapours.

This work, as well as other few theoretical works on the issue, did not invoke much interest, and stimulated no corresponding experiments, because the practical result – obtaining pulses near fixed transition frequencies – was of little practical value. In those years, it was possible to obtain frequency-tunable radiation using other methods. Many reviews in domestic and foreign literature are devoted to these methods (see, e.g., [27]). Note that they did not mention the possible use of SIT for achieving a mode locking regime.

This trend did not prevent the emergence of individual papers in the field. In 1997, a theoretical paper [39] appeared, in fact, inspired by an interest in solitons in nonlinear fibre optics. The author has coined the term 'coherent mode locking' (CML). It was clearly stated that both the amplifier and the absorber in a laser can operate in coherent regime. This removed the fundamental limitation on the pulse duration imposed by the absorption and gain linewidths in conventional lasers with a saturable absorber. In this case, a π pulse propagates in the amplifier, and a 2π pulse propagates in the absorber, which requires a twofold difference in the transition dipole moments between the amplifier and the absorber. (For the sake of fairness, we note that this idea of sequential arrangement of the amplifier and absorber in the coherent amplification regime was put forward already in a review by Kryukov and Letokhov in 1969 [12].) The possibility of mode locking in solid-state lasers as a result of the formation of SIT 2π pulses in a coherent absorber was considered theoretically also in [40, 41].

In Ref. [39], a mixture of absorbing and amplifying media was theoretically analysed in accordance with attempts to implement it experimentally in fibre lasers. In subsequent papers, the possibility of CML in quantum-cascade lasers was analysed [42, 43].

At the same time, a new line of research in the physics of solitons, devoted to 'dissipative solitons', was being formed. The concept of dissipative solitons implied such objects, whose localisation and shape in space and time is determined by a medium with gain-compensated energy losses [44]. For simplicity of the theoretical description, gain and absorption are assumed to be mixed and uniformly distributed in space. In part, such a model corresponds to a laser with an amplifier and an absorber, if we neglect the discreteness of losses and the separation of media in space. From this point of view, mode-locked lasing pulses can be regarded as dissipative solitons if their length and separation between them are much less than the laser cavity length [44].

The appearance, albeit infrequent, of theoretical works on CML did not invoke enthusiasm among experimenters. This fact, in our opinion, was facilitated by the following circumstances. There is an established belief that shortening the pulse width requires an extremely wide amplifier gain bandwidth. At the same time, very short pulses require a saturable absorber with an extremely short relaxation time. There was no understanding that in coherent interaction, when, e.g., a SIT pulse is formed, its spectrum is wider than the transition line. Coherent interaction takes place if the pulse duration is shorter than the polarisation relaxation time. In this case, absorption and amplification are observed outside the spectral line of the resonance transition. Besides, SIT experiments are relatively complex, and theoretical work that uses simplifications does not seem entirely convincing to experimenters. It was also assumed in the aforementioned papers that the self-start of lasing is impossible and the injection of seed radiation is required, which additionally complicates the experiment.

The simplification, which came from the theory of dissipative solitons, is based on mixing the absorbing and amplifying media. In contrast to this, in our papers [45, 46], we considered a laser with an amplifier, an absorber, and discrete losses on the mirrors, all spaced apart. In the case of both a linear cavity, where counterpropagating waves are present, and a ring cavity, it was shown using numerical calculations that the CML regime is possible. In a subsequent paper [47], the CML regime was analysed graphically using the McCall and Hahn 'area theorem' diagrams. In these calculations, no injection of external radiation was required, which is extremely important in practice.

Our theoretical studies convinced us of the possibility of an experimental demonstration of the CML regime. However, the first attempt was not entirely successful [48, 49]. In a cw dye laser, we placed a cell with molecular iodine vapour. By tuning the laser wavelength to an absorption line region (there are many such lines), we hoped to obtain mode locking with 2π SIT pulses. Indeed, the mode locking regime of lasing was achieved, but the pulses were not 2π SIT pulses. In our opinion, we obtained 0π pulses [49], which, apparently, are not of practical interest. Nevertheless, the lasing regime arose due to coherent interaction with the absorber. Based on these results, it is difficult to assess why exactly such pulses were generated and whether the generation of 2π pulses can occur at all.

For the first time, SIT in the form of 2π pulses in a medium located in a laser cavity was experimentally demonstrated by Diels et al. [50]. However, in this experiment, the appearance of the mode locking regime was not caused by SIT. Indeed, in Ref. [50], a nonlinear saturable absorber SESAM provided the laser mode locking, and then the lasing frequency was tuned to the absorption line of rubidium isotopes.

The demonstration of 2π pulses in Ref. [50] only in the presence of an additional absorber 'swinging' the mode locking regime also did not evidence in favour of the simplicity of the experimental implementation of the theoretical ideas of using SIT for the realisation of the mode locking regime. Nevertheless, we decided to carry out experiments with a Ti:sapphire laser, in the cavity of which, first, a cell with rubidium vapour [51–53] was placed, and then with caesium vapour. We were able to prove that the CML regime due to SIT with 2π pulses can be obtained experimentally both in a linear and in a ring cavity (details are described in the mentioned publications). The results of these studies are summarised in reviews [54, 55]. The regime is self-starting, and, what is important, the pulse duration decreases with increasing lasing power. This distinguishes the regime from the corresponding one in a laser with a saturable absorber. In this case, the structure of the sublevels, inhomogeneous broadening, and other factors did not prevent the appearance of SIT, since the pulse durations were shorter than the corresponding relaxation times.

The pulse durations obtained in experimental studies cannot yet compete with the values achieved in a similar Ti:sapphire laser with a Kerr lens. However, these studies demonstrated important features of the CML, namely, self-start of the lasing regime and reduction in the pulse duration proportional to the generation power. The latter circumstance, we hope, will allow obtaining extremely short single-cycle pulses in the CML regime with a further increase in the pump. This was pointed out in the papers [56, 57], which became a natural continuation, on the one hand, of the studies on CML in the regime of multi-cycle pulses, and on the other hand, a series of papers on extremely short dissipative SIT solitons in a cavityless scheme, which is a matrix with embedded active (with laser gain) and passive centres [58–61]. Indeed, for lasers with a long cavity, as in Refs [56, 57], the longitudinal mode spectrum becomes almost continuous and the CML pulses coincide with the dissipative solitons of the cavityless scheme. These studies are part of the recently intensively developing branch of nonlinear optics, devoted to extremely short, down to unipolar, radiation pulses, which have many unusual and attractive properties for applications (see reviews [62, 63] and monograph [64]).

As emphasised in these publications, instead of the area of the pulse envelope in the McCall and Hahn theory of self-induced transparency in multicycle pulses [8, 9], for extremely short pulses, the key role is played by the electric area of the pulse $\boldsymbol{S}_E = \int \boldsymbol{E} \mathrm{d}t$, where $\boldsymbol{E}$ is the electric field strength, and t is the time. Its value is maximum for unipolar pulses, for which the action of the field on free or bound charges is unidirectional and therefore most efficient. Earlier in Ref. [65], stationary unipolar SIT pulses were found theoretically in a cavityless scheme – a medium of passive two-level atoms (neglecting weak absorption); however, it remained unclear how they could be obtained from the initial 'standard' bipolar pulses. The introduction of a dissipative factor (active atoms), as in the papers mentioned above, allowed creating an approach to solving this problem.

One more feature of the coherent mode locking regime was demonstrated in Ref. [66], namely, a reduction in the pulse duration inevitably requires a decrease in the cavity length. This conclusion was made based on the derived similarity rule for mode-locked lasers. It is interesting due to a possibility to extrapolate the results of numerical calculations for one set of parameters to situations with the laser parameters significantly changed according to certain rules.

The requirement of reducing the cavity length to decrease the pulse duration was also confirmed in our experiments.

With an increase in the lasing power, the duration decreases to a certain value, but then harmonic regimes of mode locking arise.

In our opinion, the generality of the existing theoretical concepts and experimental results allow serious consideration of the real practical implementation of extremely short pulses in the CML regime. A laser with such properties will have an extremely short cavity (about ten wavelengths), and its implementation will become a difficult technological task that requires a selection of media and a solution to the problem of thermal and radiation resistance of materials. However, the expended efforts and funds will allow obtaining a miniature source of extremely short pulses with a terahertz repetition rate. Perhaps, this will allow increasing the speed of information transfer and processing by orders of magnitude. It is also possible that they will be used as master oscillators for obtaining even shorter unipolar pulses, the interest in which has been recently growing [63].

In experiments with a Ti:sapphire laser, it was also possible to observe extremely rare events of a sharp increase in the amplitude and duration shortening of the pulses arising in the CML regime. Recently, interest in unexpected natural phenomena has increased. This was initiated by research in oceanography of ocean waves of anomalously large amplitude, which pose a threat to shipping and coasts [67]. In the process of studying the mechanisms of their origin, it became clear that there are interesting analogies between rare events of different physical nature, in particular, between oceanography and optics [68, 69].

If we draw an analogy with a pool of water, then outside the area of an extreme event, many waves create a pattern similar to chaotic ripples on the surface of the water. Then the ripples begin to disappear, and small waves very quickly merge into one wave of large amplitude. This wave arises quickly and disappears just as quickly. Finally, ripples of waves moving in opposite directions remain on the surface without any visible order. The nature of such events is statistical, but the probability of their occurrence is many orders of magnitude greater than in the case of many independent linear waves.

We succeeded in expanding the range of such phenomena, namely, we have found [70] that such extreme events can be observed in a Ti:sapphire laser, in the cavity of which 2π pulses are present. The latter can be interpreted as dissipative SIT solitons [64]. In the presence of several pulses that demonstrated interaction with each other, formation of soliton molecules, merging of several solitons into one, and decay of solitons were recorded. The events of merging of all solitons into one were extremely rare; they were accompanied by a sharp increase in the peak power of the laser pulse and a decrease in its duration. Within the framework of the current terminology, such an event is classified as extreme. Extreme events are a consequence of the internal dynamics of the laser rather than fluctuations of a technical nature. Until now, extreme phenomena in systems of dissipative SIT solitons have not been predicted or observed experimentally.

Figure 1 shows an example oscillogram of an extreme event in a laser with a caesium vapour cell generating the radiation at the D2 transition wavelength (852 nm). The extreme event is recorded in the centre of the oscillogram. The interval between such events under the conditions of the experiment was, on average, about 5 s.

Figure 2 shows the result of ‘cutting’ the oscillogram of Fig. 1 into fragments with the duration corresponding to the double time of the cavity round trip (vertical axis). The frames are stacked along the horizontal axis. Such frame-by-frame processing of the oscillogram shows the presence of a soliton gas, i.e., a set of many solitons beyond the extreme event.

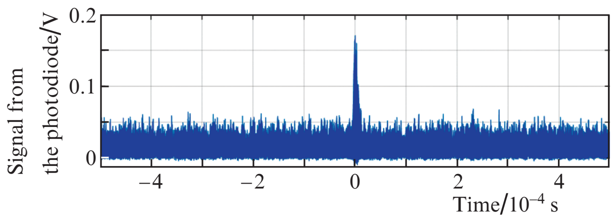

Figure 1. Example of an oscillogram containing an extreme event.

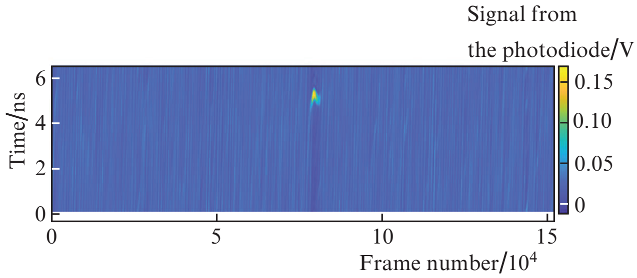

Figure 2. (Colour online) Frame-by-frame processing of the oscillogram.

The behaviour of a soliton gas near an extreme event is shown in Fig. 3. One can see the stages of solitons merging, the emergence of one intense soliton and its subsequent decay.

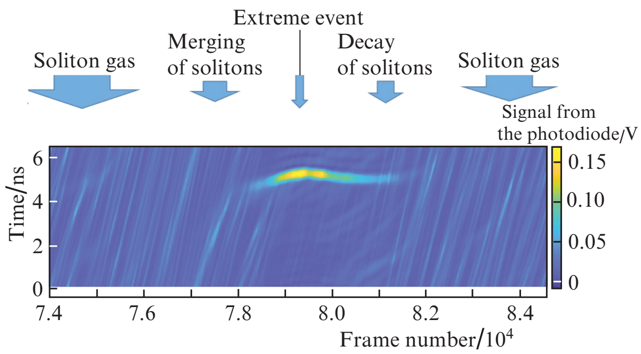

Figure 3. (Colour online) Magnified fragment of Fig. 2 near an extreme event. At the top, from left to right, arrows show the soliton gas areas; the process of merging of solitons; the extreme event of a soliton of extreme amplitude followed by its decay and the appearance of a soliton gas.

To conclude the section, let us return to the possible ‘out-of-cavity’ applications of SIT for obtaining extremely short pulses and for compressing light pulses. Whereas in [21] the possibility of compressing multicycle laser pulses due to the SIT phenomenon was shown, in the recent paper [71] a scheme of a multistage compressor of single-cycle pulses was proposed. As follows from the calculations, the unipolar half-waves of opposite polarity constituting a single-cycle pulse can behave as separate pulses and, as a result of propagation at different speeds in the SIT regime, they can approach each other. This leads to compression and significant frequency shift of the original single-cycle pulse.

However, under certain conditions, the opposite scenario is also possible, in which the SIT half-waves, on the contrary, repel each other [72]. In this case, two unipolar pulses of opposite polarity, separated in time, appear at the exit from the medium. This mechanism makes it possible to obtain unipolar half-cycle attosecond pulses with a large electric area, which is important for practical applications, such as efficient control of wave packet dynamics in various substances, acceleration of charged particles, etc. [73–79]. It seems that the SIT phenomenon will most clearly manifest its potential for practical applications of extremely short pulses and for essentially dissipative (laser) systems.

3. Topological dissipative optical solitons

Let us clarify that by the term 'optical soliton' we mean a stable localised structure of radiation in a nonlinear medium or system, where the localisation is caused by nonlinear factors, and the size of the structure is weakly related to the scale of the medium inhomogeneity. In an ideal situation, the medium or system is homogeneous in the directions of nonlinear localisation and the position of the soliton in these directions is arbitrary (determined by the initial conditions).

In this Section, we will consider, as in the traditional formulation of nonlinear optical problems, the radiation regimes close to a quasi-monochromatic and quasi-plane wave, but with a complex internal structure. Namely, the radiation will here have singularities of one or two types [80–85]. The first type manifests itself already for scalar waves, which in optics corresponds to a significant value of only one of the field polarisation components. The singularity is the uncertainty in the phase of the wave; for polarised radiation, one can speak of singularities (dislocations) of the wavefront of the polarisation components separately. In a fixed cross section, the only one allowed for two-dimensional solitons, dislocations can be isolated (individual points of the cross section) or non-isolated (lines or areas). Isolated (screw) wavefront dislocations of a scalar structure or a polarisation component of a vector structure are characterised by an integer topological charge m, i.e., the phase incursion when traversing the dislocation along a closed contour in the beam cross section divided by 2π. Below we will also give an example of a non-isolated (edge) dislocation of a wavefront. For nonzero topological charges, the energy flux in the vicinity of screw phase dislocations has a vortex character. The second type is generally possible only for polarised radiation with elliptical polarisation. Here, singularities are distinguished with purely circular polarisation, when the concept of the major and minor axes of the polarisation ellipse (a set of C-points in space) is not defined, and with purely linear polarisation when the direction of rotation of the end of the electric intensity vector along the ellipse is uncertain in time (L-points). The topological index (Poincaré index η) of isolated polarisation singularities is equal to the number of revolutions of the polarisation vector when traversing the singular point along the same contour as in the case of phase singularities [80]. Taking into account the fact that theoretical and experimental studies of dissipative optical solitons are described in many monographs [86–96, 44, 64], below we present a review of the properties of such solitons with phase and polarisation singularities, together with a number of original results. To focus precisely on these singularities, we restrict ourselves here to considering Class A lasers or laser media with a fast saturable absorber; an analysis of the manifestations of relaxation processes can be found in paper [64] and references therein.

3.1. Dissipative optical solitons with phase singularities

One of the simplest schemes in which dissipative optical solitons arise is a wide-aperture laser with a saturable absorber, which generates a single longitudinal mode [97]. For a transversely two-dimensional laser, the equation for the envelope E of the electric field strength with predominantly linear polarisation in the quasi-optical approximation and the mean-field approximation (averaging the envelope in the longitudinal direction) [98] has a dimensionless form:

$$\frac{\partial E}{\partial t} - (\mathrm{i} + d)\Delta_\perp E = f(|E|^2)E. \quad (1)$$

Here $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplace operator; and x and y are transverse coordinates orthogonal to the cavity axis z. The term with the Laplacian describes the diffraction of radiation (coefficient i) and the angular selectivity of losses (coefficient d, $0 < d \ll 1$, losses for the axial direction of propagation are minimal). We emphasise that the presence of the angular selectivity of losses is essential for ensuring the stability of topological solitons. The function f of intensity $I = |E|^2$ at small frequency detunings is real and can be written in the form:

$$f(I) = -1 + \frac{g_0}{1+I} - \frac{a_0}{1+bI}. \quad (2)$$

The first term on the right-hand side of Eqn (2) represents nonresonant absorption; g_0 and a_0 are the linear gain and absorption coefficients; and b is the ratio of the gain and absorption saturation intensities. This function reflects the balance of the inflow (gain) and outflow (absorption) of radiation energy. Laser (dissipative) solitons exist under conditions of bistability of transversely homogeneous regimes, when the non-lasing regime $E = 0$, which is realised at $f(0) < 0$, and the regime with positive intensity are stable [the equation $f(I) = 0$ has two positive roots].

Under these conditions, in a certain range of parameters, Eqn (1) has stable solutions corresponding to an axisymmetric localised intensity distribution and various values of the topological charge m. In polar coordinates (r, φ) ($x = r\cos\varphi$, $y = r\sin\varphi$)

$$E(r, \varphi, t) = A(r)\exp(\mathrm{i}m\varphi - \mathrm{i}\nu t). \quad (3)$$

The radiation is monochromatic, the frequency shift ν serves as an eigenvalue of the boundary value problem (1) with the boundary condition $E(r, \varphi, t) \to 0$ at $r \to \infty$. The spectrum of the axisymmetric problem is discrete, which corresponds to the calibrated character of dissipative solitons.

A complete analysis of such solitons is presented in [64], where references to original papers are also given. In the case under consideration, the phase singularities are located at isolated points of the laser cross section. When going beyond the mean-field approximation and taking into account the longitudinal variation of the envelope in a laser with a cavity length L, Eqn (1) is replaced by the expression:

$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial z} - (\mathrm{i} + d_{\perp})\Delta_{\perp} E - (\mathrm{i} + d_{\parallel}) D_2 \frac{\partial^2 E}{\partial t^2} = f(|E|^2)E\,. \quad (4)$$

Here D_2 is the coefficient of quadratic dispersion. For a ring resonator, an additional condition is set for the periodicity of the total field strength with a period L. The 'tubular' solitons arising in this case were analysed in Ref. [99]. Wavefront dislocations are located on vortex lines in three-dimensional space. These lines can be considered to be oriented in accordance with the direction of the phase increase in their vicinity (the phase is not defined on the lines themselves). At subcritical cavity lengths, $L < L_{\mathrm{cr}}$, vortex lines are straight, including in the case of multiple dislocations of a two-dimensional soliton with a topological charge $|m| > 1$. At a greater length L, the vortex lines are bent (Fig. 4a) and, moreover, in the case of initial m-fold dislocations, they split into $|m|$ lines with a unit topological charge (Fig. 4b).

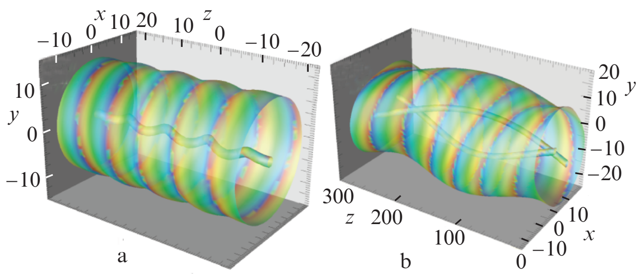

Figure 4. (Colour online) Surfaces of equal intensity in tubular solitons showing bending (a, $m = 1$) and splitting (b, $m = 2$) of vortex lines of tubular solitons at the supercritical cavity length. Here and in Figs 6 and 7, the coordinates x, y, and z are measured in laser wavelengths.

Such tubular solitons are transitional between two-dimensional and three-dimensional ones. At large cavity lengths, the presence of its mirrors becomes unimportant, and one can speak of three-dimensional solitons in an unbounded laser medium (with saturable absorption). An extensive class of vortex three-dimensional laser solitons was found in Refs [100–104] and in the papers referenced in [64]. On the vortex line itself, the energy flux vanishes, but around it, the flux component along the tangent to the line may have the same or different direction along the entire line. In the first case, the vortex line is called constant-sign, in the second case sign-changing. The number of direction changes is one of the characteristics of the vortex line. The set of all vortex lines of these solitons with unit topological charges forms their 'skeletons' of various shapes. As can be seen from Fig. 5, these lines can be closed (with a finite length) and open (with an infinite length in an unlimited space or ending at mirrors in a finite-length cavity), connected and disconnected, knotless or knotted with different topological indices (knot theory is presented, for example, in Ref. [105]). The use of this class of three-dimensional solitons significantly expands the possibilities of coding topologically protected information. In addition to topological characteristics, information can be encoded by the above number of changes in the direction of the energy flow.

3.2. Solitons with polarisation singularities

3.2.1. Singularities of 2D laser solitons. The dynamics of many wide-aperture lasers is described in the mean-field approximation by equations for the two-dimensional distribution of the field envelope and material equations for the medium. Considering polarisation, the appropriate equations for the field are two quasi-optical equations for the Cartesian components of the vector envelope $\boldsymbol{E} = \{E_x, E_y\}$ or for the amplitudes of circular polarisation $E_{\pm} = (1/\sqrt{2})(E_x \pm \mathrm{i}E_y)$. The medium dynamics for vertical-cavity semiconductor lasers is sufficiently well described by the spin-flip model equations [106]. The initially four-level model in the inertialess approximation effectively turns into a two-level one. As a result, for a laser with saturable absorption, we arrive at a simplified system of dimensionless equations

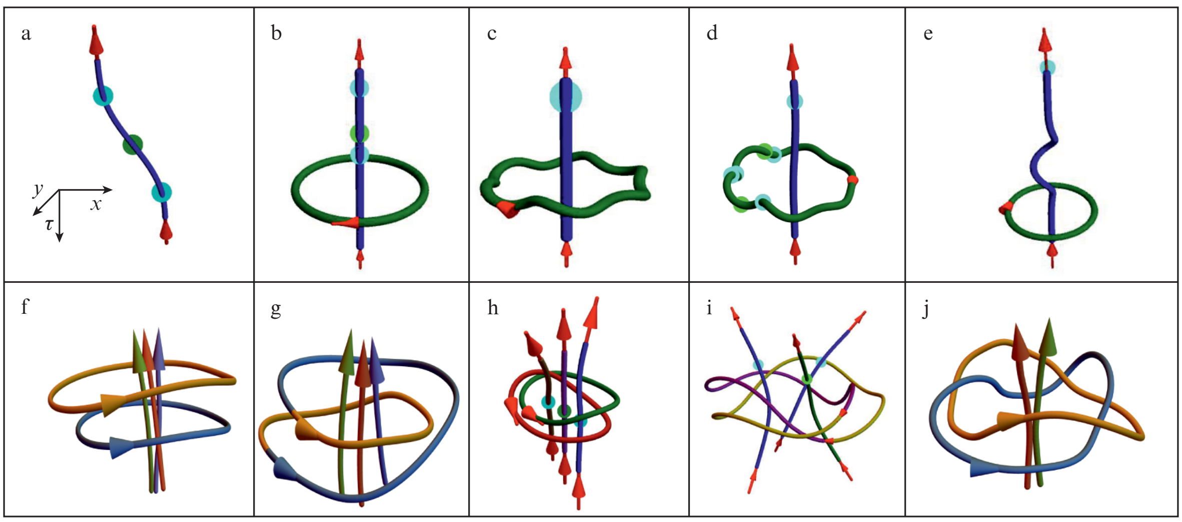

Figure 5. (Colour online) Vortex lines of 3D laser solitons. The number of closed vortex lines is 0 (a), 1 (b–e, g, i), and 2 (f–h, k); the trivial knot (g) and the trefoil knot (k) are presented. The number of open lines is 1 (a–e), 2 (k) and 3 (f–i). The arrows on the vortex lines indicate the direction of the increase in the radiation phase in the vicinity of the line, circles on them (a–d, h, f) mark the places where the direction of the energy flux component changes along the tangent to the vortex line.

$$\partial_t E_\pm - (\mathrm{i} + d)\nabla_\perp^2 E_\pm = f(I)E_\pm,$$
$$f(I) = -1 - \frac{a_0}{1+I} + \frac{(1-\mathrm{i}\alpha)g_0}{1+bI}, \quad (5)$$

where $I = |E_+|^2 + |E_-|^2 = |E_x|^2 + |E_y|^2$ is the total intensity; and α is the α-factor characterising semiconductor lasers. In the Cartesian components

$$\partial_t E_{x,y} - (\mathrm{i} + d)\nabla_\perp^2 E_{x,y} = f(|E_x|^2 + |E_y|^2)E_{x,y}. \quad (6)$$

In a more general case, the function f depends not only on the sum, but also on the difference in the intensities of the polarisation components, and allows for fast population oscillations, which are of no fundamental importance for this consideration [106, 107]. Further, following [108], we will trace the polarisation structure of dissipative solitons that follows from Eqns (5) and (6).

Obviously, these equations are invariant with respect to the phase shifts of each of the polarisation components. In addition, they reduce to a single equation describing a scalar structure with a total intensity I if the two polarisation components have the same intensity distribution, but different phase distributions (see examples below). In this case, we can obtain one of the polarisation components by multiplying the scalar distribution by a factor q, and the other component by a factor $\sqrt{1-q^2}$, where $q = \mathrm{const}$, and $0 \leqslant q^2 \leqslant 1$ (the difference in phase distributions should be additionally taken into account). Consequently, we have a family of solitons with a continuously varying parameter q (continuous spectrum).

If we apply this reasoning to the initial scalar soliton, for which the polarisation state is the same over the entire cross section, we obtain

$$E_+ = qE_0, \; E_- = \sqrt{1-q^2}\,E_0, \; q = \mathrm{const}. \quad (7)$$

Accordingly, $E_+/E_- = q/\sqrt{1-q^2} = \mathrm{const}$. Thus, in addition to phase factors, there is a one-parameter family of solitons with the same polarisation state generated by a scalar soliton.

At equal intensities of circular polarisation components ($q^2 = 1/2$), when $E_+ = \pm E_-$ (the case of coincidence of the phase distributions of the components), the amplitude of one Cartesian polarisation component vanishes over the entire laser aperture, i.e., the electric field strength is everywhere directed along the other axis of the Cartesian coordinates (linear polarisation). At $q = 0$ or ± 1, the polarisation is purely circular, and the intermediate values of the parameter correspond to elliptical polarisation.

More interesting is the case when the circular field components correspond to scalar solitons with different phase distributions, but with the same intensity distributions. This is possible, e.g., if these components correspond to scalar symmetric vortex solitons with opposite signs of the topological charge. For this choice with $q^2 = 1/2$ we have

$$E_\pm(x,y) = \frac{1}{\sqrt{2}}(E_x \pm E_y) = \exp(\pm\mathrm{i}\eta\varphi)A_\eta(r), \quad (8)$$

where $A_\eta(r)$ is the complex amplitude of the radial profile of scalar symmetric solitons with topological charges $m = \pm\eta$, which is a solution to the equation

$$(\mathrm{i} + d)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\eta^2}{r^2}\right)A_\eta(r) + f(|A_\eta|^2)A_\eta(r) = 0. \quad (9)$$

From (8) the form of the Cartesian polarisation components follows:

$$E_x = \sqrt{2}\,A_\eta(r)\cos(\eta\varphi), \; E_y = \sqrt{2}\,A_\eta(r)\sin(\eta\varphi), \quad (10)$$

i.e., the polarisation is purely radial with the Poincaré index η. The stability region coincides with that of the scalar soliton, since the latter does not depend on the sign of the topological charge. As for the Cartesian components have zero amplitude not at isolated points, but on the entire coordinate axes. This degenerate case can be classified as an edge dislocation. It is illustrated in Figs 6a and 6b.

The degeneracy of the phase singularities is removed at different intensities of the components ($q^2 \neq 1/2$). As can be seen from Figs 6c and 6d, in this case there is a single vortex dislocation at the centre of the structure. The polarisation singularity with the Poincaré index $\eta = 1$ is preserved.

It follows from the calculations that these structures are stable with respect to small perturbations. If at the initial moment the total intensity of the components differs from the scalar soliton intensity, then this difference disappears in a short period of time, after which a vector singular soliton is established, the stability of which is realised in the same range of parameters as for the scalar soliton.

3.2.2. Singularities of tubular solitons. Tubular solitons were considered in Section 3.1 in the paraxial approximation, in which the radiation was linearly polarised (only the field strength component E_x is significant). Beyond this approximation, which is valid under the condition that the parameter $k_0 w$ is small, where k_0 is the wave number in a linear medium and w is the characteristic width of the structure, corrections appear that allow for the more complex polarisation state of radiation. For two-dimensional laser solitons, such a weakly non-paraxial analysis was performed in Ref. [109], and for tubular solitons, in Ref. [99]. Further, based on these papers, we will present the polarisation structure of tubular solitons in more detail. For this purpose, let us consider the symmetric tubular soliton presented in Section 3.1, which is stable in a cavity with a subcritical length.

In the initial paraxial approximation in the cylindrical coordinate system, the envelope of the soliton has the form [see (3)]

$$E_x(r,\varphi,t)A(r,\varphi)\exp(-\mathrm{i}K_m z - \mathrm{i}\nu t),$$
$$A(r,\varphi) = a(r)\exp[\mathrm{i}\Phi_x(r,\varphi)], \quad (11)$$
$$\Phi_x(r,\varphi) = \Phi_x(r) + m\varphi,$$

where ν is the nonlinear frequency shift; K_m is the shift of the wavenumber; and $A(r,\varphi)$ is the transverse profile of the field envelope. In the first order with respect to the small parameter $k_0 w$, only the longitudinal component of the envelope arises,

$$E_z(r,\varphi,z) \approx \frac{\mathrm{i}}{k_0}\mathrm{div}_\perp \boldsymbol{E}_\perp = \frac{\mathrm{i}}{k_0}\exp(-\mathrm{i}K_m z)\frac{\partial}{\partial x}A(r,\varphi)$$
$$= \frac{\mathrm{i}}{k_0}A(r,\varphi)[\gamma(r,\varphi) + \mathrm{i}k(r,\varphi)]. \quad (12)$$

The following notation is used here:

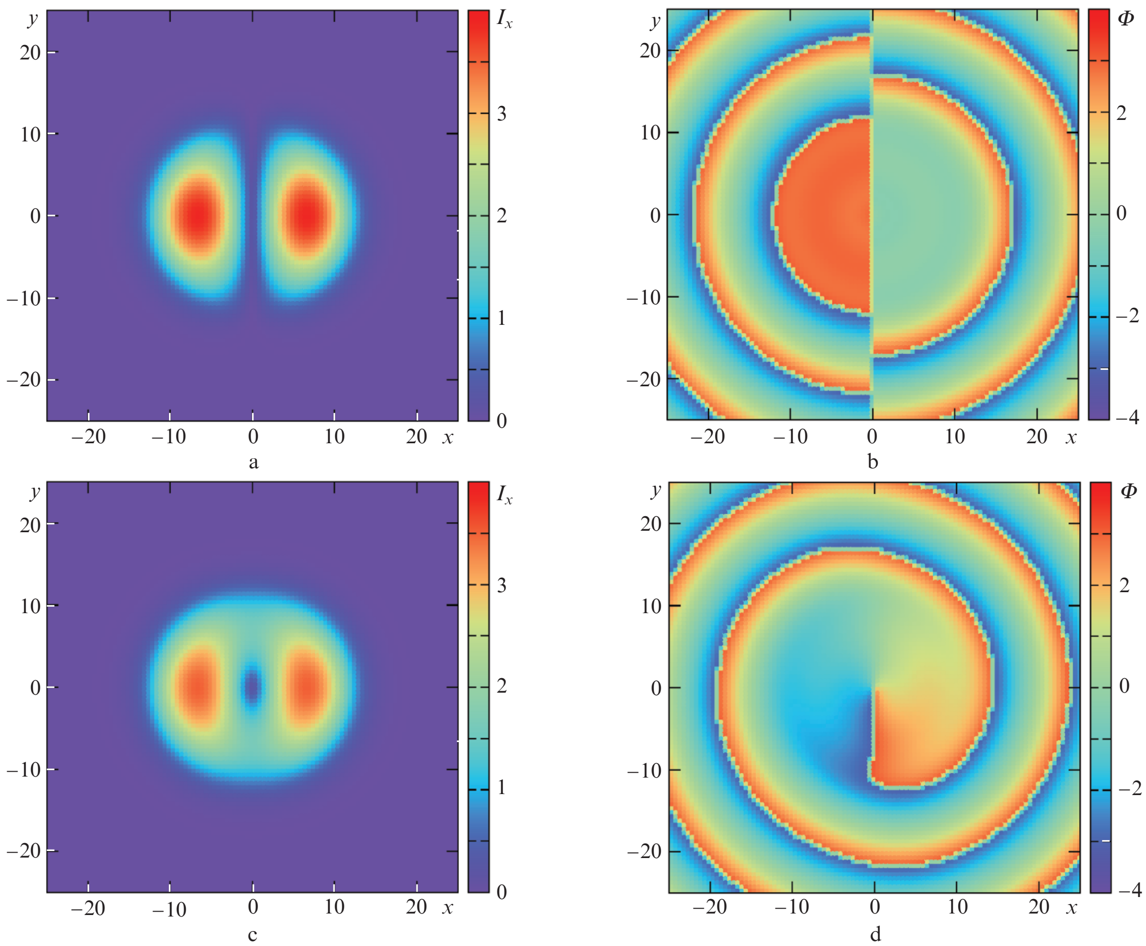

Figure 6. (Colour online) Intensity distributions of the Cartesian component $I_x = |E_x|^2$ (a, c) and its phase Φ (b, d) at $q^2 = 1/2$ (a, b) and $q^2 = 0.22$ (c, d); $\eta = 1$, $\alpha = 0$.

$$\gamma(r,\varphi) = a^{-1}(r)\frac{\partial}{\partial x}a(r) = \gamma(r)\cos\varphi\,,$$

$$k(r,\varphi) = \frac{\partial}{\partial x}\Phi_x(r,\varphi) = \frac{1}{r}[rk(r)\cos\varphi - m\sin\varphi]\,, \qquad (13)$$

$$\gamma(r) = a^{-1}(r)a'(r)\,, \quad k(r) = \Phi'_x(r)\,.$$

Thus, the polarisation state is elliptical and variates over the cross section. The electric strength vector oscillates in the plane (E_x, E_z); an exit from this plane is associated with the appearance of the field strength component E_y, but it is small, since arises only in the next order with respect to the small parameter $k_0 w$, which we do not consider here. The characteristics of the polarisation ellipses are specified by the relative phase $\delta\phi(r,\varphi) = \Phi_z(r,\varphi) - \Phi_x(r,\varphi)$ [110], determined by the relation

$$\sin(\delta\phi) = \frac{r\gamma(r)\cos\varphi}{\sqrt{[r\gamma(r)\cos\varphi]^2 + [rk(r) - m\sin\varphi]^2}}\,. \qquad (14)$$

Elliptical polarisation degenerates into linear when the phase difference $\delta\phi = \pi s$, where s is an integer. Relation (14) makes it possible to find L-lines in the cross section, on which the polarisation is linear. Indeed, the condition $r\gamma(r)\cos\varphi = 0$ is satisfied, first, on the circle on which the intensity of the paraxial soliton is maximum, $\gamma(r) = 0$, and on the y axis, orthogonal to the polarisation direction of this soliton, $\cos\varphi = 0$ (Fig. 7). These two L-lines divide the cross section into four regions with the same chirality (direction of the polarisation vector rotation), depending on the positive or negative value of $\mathrm{sign}[\gamma(r)\cos\varphi] = \pm 1$.

The longitudinal component of the electric vector reaches zero only at two points at the intersection of the circle corresponding to the maximum intensity of the paraxial soliton and the $y = 0$ axis (red horizontal arrows). At the centre of the soliton, the entire vector of electrical field strength is zero. When crossing the L-lines, the chirality is reversed. As the distance from these lines increases, the slope of the polarisation ellipse in the plane (E_x, E_z) also increases, and the slope of the major axis of the ellipse with respect to the x-axis changes its sign abruptly when crossing the L-line $x = 0$. Thus, the considered tubular solitons have screw wavefront dislocations and polarisation singularities in the form of L-lines.

4. Conclusions

Thus, the involvement of dissipative factors significantly expands the possibilities of achieving both stable extremely short radiation pulses and regimes with complex topology. Here we have illustrated this by the examples related to laser schemes.

Self-induced transparency, discovered at the very beginning of the laser era, belongs to the group of coherent nonlinear phenomena. It is of great physical importance for under-

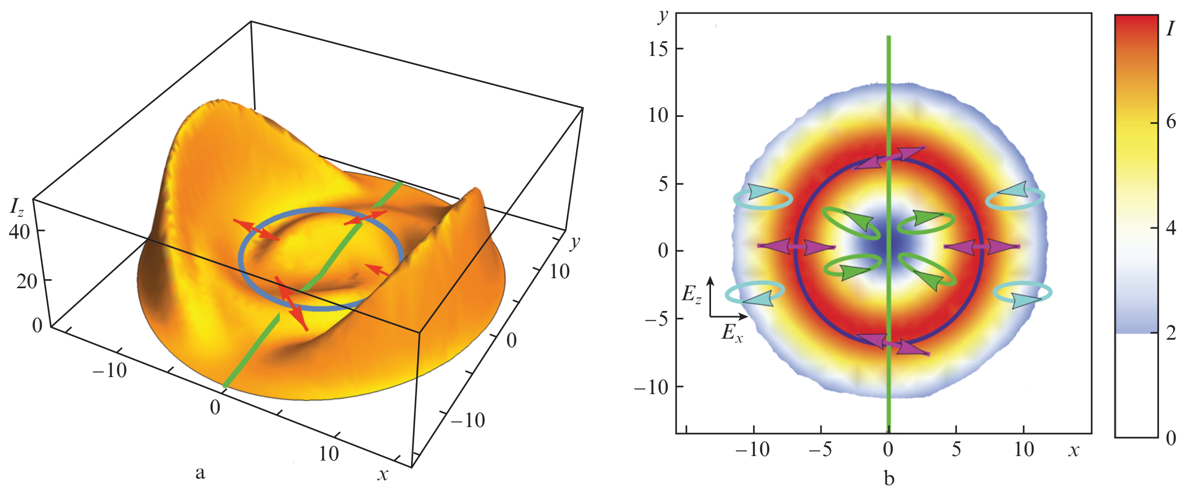

Figure 7. (Colour online) Ellipses of the polarisation vector at different points of the vortex soliton with $m = 1$. The intensity distribution in the (x, y) plane corresponds to the colour scale. The electric field strength vector changes in the plane (E_x, E_z). The arrow on the ellipses shows the direction of rotation of the electric field strength vector in time.

standing the mechanism of the interaction of matter with radiation, since it demonstrates those properties of the interaction, in which the coherence of the states of matter plays an important role. At the same time, this phenomenon has not yet found practical application. One of the seemingly possible applications of the phenomenon was associated with attempts to use the SIT regime in an absorbing medium for mode locking in lasers. However, the principles and technologies of modulating cavity losses due by means of saturable absorbers and practically inertialess Kerr lenses turned out to be more universal and easy to use. Nevertheless, such methods of passive mode locking in lasers have a number of fundamentally unavoidable drawbacks. For instance, the duration and repetition period of pulses in lasers are limited by the width of the gain lines, and in lasers with a saturable absorber, by the width of the absorption line. These limitations, which prevent the production of extremely short radiation pulses, can be overcome only in the regime of coherent mode locking. Its practical implementation can lead to the creation of a new generation of compact sources of single-cycle pulses with a terahertz repetition rate.

It is possible that the creation of such lasers will be hindered by some, so far unknown, difficulties. Nevertheless, the SIT effect itself inside the laser again became interesting from an academic point of view. In a laser, regimes with harmonic mode locking are possible, when many SIT pulses are simultaneously present in the cavity. Such a system can be regarded as a gas of dissipative solitons. In such a gas, as our recent experiments have shown, soliton molecules arise, and processes of fusion and decay of dissipative SIT solitons take place. The extreme phenomenon of ‘rogue waves’ discovered by us has recently become a subject of growing interest. Thus, the nonlinear dynamics of solitons of various origin is enriched with new aspects that manifest themselves in studies of dissipative laser SIT solitons. The possibility of creating a multistage optical compressor based on the SIT phenomenon also attracts attention. The prospects that open up are consonant with the tendencies of modern nonlinear optics and laser physics to achieve ever shorter radiation pulses.

Another section of nonlinear optics discussed above reveals the extremely rich internal structure of multidimensional dissipative optical solitons associated with the topology of both phase and polarisation singularities. These singularities, due to their topological protection, are promising for information applications. The dissipativity of the laser systems considered adds new aspects associated with the nontrivial structure of energy fluxes.

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