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Modulation instability of two TE modes in a thin left-handed film on a nonlinear right-handed substrate

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Abstract. **The intramode wave beams in a thin left-handed film on a Kerr substrate are considered at a frequency near zero mode group velocity. Four coupled (1 + 1)-dimensional nonlinear Schrödinger equations, describing the interaction of forward and backward propagating beams with positive and negative group velocities, are derived. It is shown that self- and cross-phase modulation for four simultaneously propagating modes is possible only at strictly matched perturbations of their propagation constants, which is due to the contribution of spatial parametric mixing. The modulation instability of only two waveguide modes is analysed for different versions of their propagation. The specific features of modulation instability, related to the propagation of modes with negative group velocities, are investigated.**

Keywords: *thin left-handed film, waveguide modes, negative group velocity, modulation instability.*

1. Introduction

Modern nanotechnologies make it possible to design left-handed metasurfaces having simultaneously negative permittivity and permeability, which are formed by a thin left-handed film on a right-handed substrate with positive permittivity and permeability [1 – 4]. Studies of the new waveguide properties of these metasurfaces in the optical wavelength range, including those caused by nonlinear optical response, are urgent for technical applications.

The nonlinear wave equation describing the propagation of an electromagnetic wave in a bulk right-handed medium with a nonlinear Kerr-type response (Kerr effect) has solutions in the form of a plane wave with a phase depending on its intensity [5 – 13]. This wave may be unstable with respect to small amplitude perturbations both in time, at a negative group-velocity (anomalous) dispersion, and in space, at a positive Kerr coefficient (in a self-focusing medium). In the latter case modulation instability leads to small-scale self-focusing of a homogeneous wave and, eventually, to its decomposition into separate beams. The presence or absence of modulation instability is closely related to the existence of bright or dark envelope solitons [10 – 13].

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Modulation instability may develop during propagation of not only plane waves but also limited laser beams in bulk nonlinear media (in particular, in nonlinear optical cavities and laser amplification systems) and during surface wave propagation in nonlinear layered media and guided modes in nonlinear planar waveguides and optical fibres [10 – 20]. Under these conditions, the instability of a guided mode with respect to small amplitude perturbations in a nonlinear optical fibre or in a planar Kerr waveguide develops in time in the case of anomalous dispersion as well [21 – 30]. In a planar waveguide with self-focusing nonlinearity, the modulation instability of one waveguide mode may also develop in space (in the waveguide plane) [10, 20, 31 – 38].

Simultaneous propagation of two guided modes in a nonlinear waveguide is also accompanied by instability of their amplitudes with respect to small perturbations. Due to the intermodal interaction their modulation instability may develop at the parameters corresponding to stable single-mode propagation [12, 13, 39 – 48].

The studies of the modulation instability in nonlinear left-handed metamaterials and waveguide structures containing these materials have provided a much deeper insight into their development conditions [49 – 62]. In some cases these conditions can be considered as inverted with respect to those for modulation instability development in right-handed media. For example, when only one wave propagates in left-handed media with a nonlinear Kerr-type response, spatial modulation instability may occur at a negative Kerr coefficient [51], and time instability is possible at a positive group-velocity dispersion [52].

The dispersion properties of the optical waveguides containing left-handed metamaterials differ dramatically from the properties of conventional right-handed waveguides [63 – 71]. Left-handed waveguides, as well as right-handed ones, allow for propagation of fast waveguide modes with a phase velocity exceeding that of plane waves in a bulk material having the same parameters as the film material. However, in contrast to right-handed waveguides, left-handed ones allow also for propagation of slow waveguide modes with a phase velocity smaller than the plane-wave phase velocity in the bulk film material. The frequency dispersion relation for the propagation constant of a fast guided mode of a planar waveguide based on a thin left-handed film, a right-handed cover medium, and a right-handed substrate may have a point at which the mode group velocity becomes zero [63, 67, 69 – 71]. This point divides the dispersion curve into two branches. One of them corresponds to either forward or backward propagating mode with a positive group velocity, having the same direction as the phase velocity, while the other branch corresponds to either forward or backward propagating mode with a negative group velocity, whose direction is opposite to that of the phase velocity. Therefore, in the general case, four guided modes of the same type may propagate at the same frequency and in the same direction in a thin left-handed film [69–71], in contrast to waveguides based on right-handed materials, which allow for propagation of only two such modes [72–76].

If the material of a left-handed waveguide (e.g., substrate [68, 71]) has a nonlinear optical response, the propagation of modes of the same type is accompanied by not only the self- and cross-phase modulation, as in conventional right-handed waveguides [10, 12, 13, 77–81], but also intermodal energy exchange [71]. The development of spatial modulation instability during propagation of guided modes in this nonlinear planar waveguide may possess essential features, which have not been considered until now. In this paper, we report the results of analysing the modulation instability for different versions of propagation of two waveguide modes in a thin left-handed film on a right-handed substrate exhibiting the Kerr effect at a frequency near zero group velocity.

2. Intramodal beams at a frequency near zero group velocity

Let us consider a planar waveguide based on a thin left-handed film and a substrate exhibiting the Kerr effect (Fig. 1). We assume that the frequency dispersion of the relative permittivity and permeability of the film metamaterial is described by the model relations used in [50–52, 54, 55, 57, 63, 67–71] and the layer parameters accepted in [69–71]. Generally, the light field in this waveguide is formed by four monochromatic beams of fast TE modes with the same mode index, exceeding unity [70, 71]. The frequency dispersion relations for the propagation constants of these modes have a point at which the mode group velocity becomes zero [63, 67, 69–71, 82]. The central component of the spatial spectrum of the first (second) beam is the forward propagating mode with a positive (negative) group velocity, to which the wave vector $\boldsymbol{\beta}_+$ ($\boldsymbol{\beta}_-$) corresponds. The central component of the spatial spectrum of the third (fourth) beam is the backward propagating mode, also with a positive (negative) group velocity, to which the wave vector $-\boldsymbol{\beta}_+$ ($-\boldsymbol{\beta}_-$) corresponds.

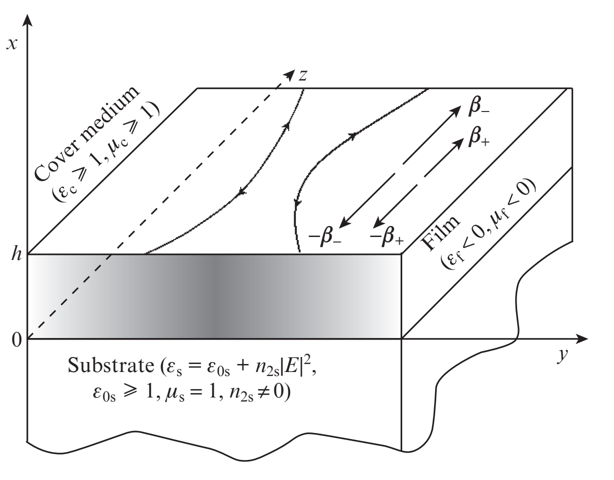

Figure 1. Schematic of a three-layer planar waveguide based on a left-handed film, a right-handed cover medium, and a right-handed substrate exhibiting the Kerr effect.

In the paraxial approximation the y component of the electric strength vector of the total light field, which is a superposition of four intramode beams, can be written as

$$E_y(x,y,z) = \sqrt{I_{\text{in}}}\,[\Psi_+(x)\,C_+^{\text{f}}(y,z)\exp(-\mathrm{i}\beta_+ z)$$

$$+\,\Psi_+(x)\,C_+^{\text{b}}(y,z)\exp(\mathrm{i}\beta_+ z) + \Psi_-(x)\,C_-^{\text{f}}(y,z)\exp(-\mathrm{i}\beta_- z)$$

$$+\,\Psi_-(x)\,C_-^{\text{b}}(y,z)\exp(\mathrm{i}\beta_- z)] + \text{c.c.}, \qquad (1)$$

where I_{in} is the maximum light field intensity in the waveguide; $\Psi_+(x)$ and $\Psi_-(x)$ are dimensionless functions, describing the spatial distribution of the mode field along the normal to the film (Fig. 1) for modes with positive (sign +) and negative (sign –) group velocities [69, 70]; β_+ and β_- are the mode propagation constants; $C_{+,-}^{\text{f,b}}(y,z)$ are dimensionless functions, describing the spatial distribution of light field envelope in the film plane (slow and fast dependences along the z and y axes, respectively); and the superscripts f and b refer to the forward and backward propagating modes, respectively.

Using the standard procedure of the method of slowly varying amplitudes and neglecting the processes that do not obey the phase-matching conditions [12, 13, 45, 72–79], one can derive (from the Maxwell equations) the following equations, describing the interaction between the considered intramodal beams in the paraxial approximation:

$$-\mathrm{i}\frac{\partial C_+^{\text{f}}}{\partial\zeta} + \frac{1}{2}\frac{\partial^2 C_+^{\text{f}}}{\partial\eta^2} + \text{sign}n_{2\text{s}}\{[|C_+^{\text{f}}|^2 + 2|C_+^{\text{b}}|^2 + 2g(|C_-^{\text{f}}|^2$$

$$+|C_-^{\text{b}}|^2)]C_+^{\text{f}} + 2g(C_+^{\text{b}})^* C_-^{\text{f}} C_-^{\text{b}}\} = 0, \qquad (2)$$

$$\mathrm{i}\frac{\partial C_+^{\text{b}}}{\partial\zeta} + \frac{1}{2}\frac{\partial^2 C_+^{\text{b}}}{\partial\eta^2} + \text{sign}n_{2\text{s}}\{[2|C_+^{\text{f}}|^2 + |C_+^{\text{b}}|^2 + 2g(|C_-^{\text{f}}|^2$$

$$+|C_-^{\text{b}}|^2)]C_+^{\text{b}} + 2g(C_+^{\text{f}})^* C_-^{\text{f}} C_-^{\text{b}}\} = 0, \qquad (3)$$

$$-\mathrm{i}\frac{\partial C_-^{\text{f}}}{\partial\zeta} + \frac{\delta_\beta}{2}\frac{\partial^2 C_-^{f}}{\partial\eta^2} - \text{sign}n_{2\text{s}}\{[2g(|C_+^{\text{f}}|^2 + |C_+^{\text{b}}|^2)$$

$$+\,g_1(|C_-^{\text{f}}|^2 + 2|C_-^{\text{b}}|^2)]C_-^{\text{f}} + 2gC_+^{\text{f}}C_+^{\text{b}}(C_-^{\text{b}})^*\} = 0, \qquad (4)$$

$$\mathrm{i}\frac{\partial C_-^{\text{b}}}{\partial\zeta} + \frac{\delta_\beta}{2}\frac{\partial^2 C_-^{\text{b}}}{\partial\eta^2} - \text{sign}n_{2\text{s}}\{[2g(|C_+^{\text{f}}|^2 + |C_+^{\text{b}}|^2)$$

$$+\,g_1(2|C_-^{\text{f}}|^2 + |C_-^{\text{b}}|^2)]C_-^{\text{b}} + 2gC_+^{\text{f}}C_+^{\text{b}}(C_-^{\text{f}})^*\} = 0, \qquad (5)$$

where $\eta = (\beta_+|\gamma|/2)^{1/2}y$ and $\zeta = |\gamma|z/2$ are normalised coordinates; $\text{sign}n_{2\text{s}}$ is the sign of the substrate Kerr coefficient; and $\delta_\beta = \beta_+/\beta_-$. As in [71], we introduced the coefficients

$$\gamma = \frac{3}{4}\frac{\pi\chi_{3s}I_{\rm in}}{\lambda n_+}\frac{J_+}{N_+},\;\; g = \frac{\beta_+}{\beta_-}\frac{N_+}{N_-}\frac{J}{J_+},$$

$$g_1 = \left(\frac{\beta_+ N_+}{\beta_- N_-}\right)^2\frac{J_-}{J_+}, \tag{6}$$

overlap integrals

$$J_+ = \int_{-\infty}^{0}\Psi_+^4(x)\,{\rm d}x,\;\; J = \int_{-\infty}^{0}\Psi_+^2(x)\,\Psi_-^2(x)\,{\rm d}x,$$

$$J_- = \int_{-\infty}^{0}\Psi_-^4(x)\,{\rm d}x \tag{7}$$

and normalisation integrals

$$N_+ = \int_{-\infty}^{\infty}\frac{\Psi_+^2(x)}{\mu(x)}{\rm d}x,$$

$$N_- = \left|\int_{-\infty}^{\infty}\frac{\Psi_+^2(x)}{\mu(x)}{\rm d}x\right|; \tag{8}$$

where λ is the light wavelength in vacuum; $n_+ = \beta_+\lambda/2\pi$ is the effective refractive index of the mode with positive group velocity; χ_{3s} is the nonlinear optical susceptibility of the substrate; and $\mu(x)$ is the dependence of relative waveguide permeability along the normal to the waveguide plane (Fig. 1). The negative value of the integral under the modulus sign in the formula for N_- is directly taken into account in Eqns (4) and (5).

The system of equations (2)–(5) is a system of coupled nonlinear Schrödinger equations (NSEs). If only one intramodal beam propagates in a waveguide (for example, a beam with a nonzero envelope $C_+^{\rm f} \neq 0$ under the condition $C_+^{\rm b} \equiv C_-^{\rm f,b} \equiv 0$), the corresponding equation [e.g., Eqn (2)] is the conventional NSE [5–38]. When only two beams propagate (for example, with envelopes $C_+^{\rm f} \neq 0$ and $C_-^{\rm f} \neq 0$ at $C_+^{\rm b} \equiv C_-^{\rm b} \equiv 0$), the system of two corresponding equations [e.g., Eqns (2) and (4)] is similar to a system of two coupled NSEs [12, 13, 39–48]. The propagation of three or more beams is described by three or more coupled NSEs [12, 13, 45, 79–81] [e.g., Eqns (2)–(4) under the conditions $C_+^{\rm f} \neq 0$, $C_+^{\rm b} \neq 0$, and $C_-^{\rm f} \neq 0$ at $C_-^{\rm b} \equiv 0$], which are derived from system (2)–(5) by equating the envelope of the absent beam to zero.

Note that in the case under consideration the interaction of four modes is due to not only the self- and cross-phase modulation [12, 13, 79–81], whose contribution is described by the first four nonlinear terms in Eqns (2)–(5), but also the parametric mixing [12, 13, 45, 77, 78], whose contribution is presented by the last nonlinear term in Eqns (2)–(5). At the same time, the propagation of only one, two, or three modes is not accompanied by this mixing. In neglect of the derivatives with respect to the y coordinate, Eqns (2)–(5) describe the interaction between modes that are unlimited in the waveguide plane; an analysis of this interaction was performed in [71], where it was shown that the contribution of spatial parametric mixing of modes leads to an efficient energy exchange between them.

The signs before the nonlinear terms in Eqns (2) and (3) with the evolution derivatives of the envelopes of beams formed by modes with a positive group velocity are opposite to the signs before similar nonlinear terms in Eqns (4) and (5) with the evolution derivatives of the envelopes of beams formed by modes with a negative group velocity. These signs are determined by the sign of the nonlinear optical coefficient of the substrate, n_{2s}. In the case of one beam with a positive group velocity and envelope $C_+^{\rm f}$ (or $C_+^{\rm b}$), under the conditions $C_+^{\rm b} \equiv C_-^{\rm f,b} \equiv 0$ (or $C_+^{\rm f} \equiv C_-^{\rm f,b} \equiv 0$), it follows from Eqn (2) [or (3)] that a positive coefficient n_{2s} corresponds to a focusing Kerr nonlinearity [31–38], while a negative coefficient corresponds to a defocusing nonlinearity. For a beam with a negative group velocity and envelope $C_-^{\rm f}$ (or $C_-^{\rm b}$), under the conditions $C_-^{\rm b} \equiv C_+^{\rm f,b} \equiv 0$ (or $C_-^{\rm f} \equiv C_+^{\rm f,b} \equiv 0$), it follows from Eqn (4) [or (5)] that, vice versa, a positive coefficient n_{2s} corresponds to a defocusing Kerr nonlinearity, whereas a negative coefficient corresponds to a focusing nonlinearity.

The efficiency of the nonlinear processes described by the NSEs (2)–(5) is determined by the coupling constant. For waveguides with an intentionally doped glass substrate with a large coefficient n_{2s} (10^{-19} m^2 W^{-1} [77, 78]), at a light intensity on the order of 10^{15} W m^{-2}, the coupling constant may reach several tens of inverse centimetres [71], which corresponds to a submillimetre effective interaction length of waveguide modes.

3. Self- and cross-phase modulation of waveguide modes

Coupled NSEs, which were analysed in [5–13], have an elementary solution in the form of plane waves with a linear (in the evolution coordinate) phase delay, proportional to wave intensities. An analysis of the formation of modulation instability of these waves determines the possibility of forming soliton envelopes of different types [10–13, 22, 27, 42, 47, 60].

In our case, an analogue of the solution in the form of plane waves is the solution corresponding to propagation of four modes in the planar waveguide under consideration, which are unlimited in the waveguide plane. This solution for coupled NSEs (2)–(5) can be written as

$$C_{+,-}^{\rm f} = \Upsilon_{+,-}^{\rm f}\exp(\mp {\rm i}b_{+,-}^{\rm f}\zeta/2),$$

$$C_{+,-}^{\rm b} = \Upsilon_{+,-}^{\rm b}\exp(\pm {\rm i}b_{+,-}^{\rm b}\zeta/2), \tag{9}$$

where the normalised perturbations of propagation constants, describing the self- and cross-phase modulation, should satisfy the relation

$$b_+^{\rm f} - b_+^{\rm b} + b_-^{\rm f} - b_-^{\rm b} = \pi n \tag{10}$$

and be expressed in terms of mode amplitudes as follows:

$$b_+^{\rm f,b}/2 = {\rm sign}\,n_{2s}\left\{(\Upsilon_+^{\rm f,b})^2 + 2(\Upsilon_+^{\rm b,f})^2 + 2g\left[(\Upsilon_-^{\rm f})^2 + (\Upsilon_-^{\rm b})^2\right]\right.$$

$$\left. +(-1)^n 2g\left(\Upsilon_+^{\rm b,f}\,\Upsilon_-^{\rm f}\,\Upsilon_-^{\rm b}/\Upsilon_+^{\rm f,b}\right)\right\}, \tag{11}$$

$$b_-^{\rm f,b}/2 = {\rm sign}\,n_{2s}2g\left[(\Upsilon_+^{\rm f})^2 + (\Upsilon_+^{\rm b})^2\right] + g_1\left[(\Upsilon_-^{\rm f,b})^2 + 2(\Upsilon_-^{\rm b,f})^2\right]$$

$$+(-1)^n 2g(\Upsilon_+^{\rm f}\Upsilon_+^{\rm b}\Upsilon_-^{\rm b,f}/\Upsilon_-^{\rm f,b}), \qquad (12)$$

here, n is an integer. Relation (10) imposes a constraint condition on the normalised mode amplitudes; this condition can be written as

$$\left[1+(-1)^n 2g\frac{\Upsilon_-^{\rm f}\Upsilon_-^{\rm b}}{\Upsilon_+^{\rm f}\Upsilon_+^{\rm b}}\right]\left[(\Upsilon_+^{\rm f})^2-(\Upsilon_+^{\rm b})^2\right]$$

$$+\left[g_1+(-1)^n 2g\frac{\Upsilon_+^{\rm f}\Upsilon_+^{\rm b}}{\Upsilon_-^{\rm f}\Upsilon_-^{\rm b}}\right]\left[(\Upsilon_-^{\rm f})^2-(\Upsilon_-^{\rm b})^2\right]=n\pi. \qquad (13)$$

Matching of the amplitudes of four modes, set by (13), is directly related to the contribution of spatial parametric mixing, to which the last nonlinear terms in Eqns (2)–(5) correspond. In the case of propagation of any three, two, or one mode in the waveguide under consideration, this matching is absent, and the perturbations of propagation constants are described by expressions (11) and (12) after equating the absent-mode amplitudes to zero.

4. Modulation instabilityof a single mode

It is known that the plane-wave solution described by formula (9) in our case may be either unstable or stable with respect to small additive perturbations of the form $a\exp[{\rm i}(K\zeta + Q\eta)] + b\exp[-{\rm i}(K\zeta + Q\eta)]$ of amplitude Υ. Similar to the case of propagation of one monochromatic wave with a constant amplitude in a Kerr medium [5–13], modulation instability of one waveguide mode (corresponding to complex values of perturbation longitudinal wavenumber K) with a positive group velocity, e.g., with an amplitude coefficient $\Upsilon_+^{\rm f}(\Upsilon_-^{\rm f}=\Upsilon_{+,-}^{\rm b}=0)$ or $\Upsilon_+^{\rm b}(\Upsilon_{+,-}^{\rm f}=\Upsilon_-^{\rm b}=0)$, may occur only at a positive Kerr coefficient of the substrate, $n_{2\rm s} > 0$, if the mode amplitude satisfies the threshold condition $(\Upsilon_+^{\rm f,b})^2 > Q^2/4$, determined by the transverse wavenumber Q. Modulation instability of a mode with a negative group velocity, e.g., with an amplitude $\Upsilon_-^{\rm f}(\Upsilon_+^{\rm f}=\Upsilon_{+,-}^{\rm b}=0)$ or $\Upsilon_-^{\rm b}(\Upsilon_{+,-}^{\rm f}=\Upsilon_+^{\rm b}=0)$ is, vice versa, possible only at a negative coefficient, $n_{2\rm s} < 0$, if the condition $(\Upsilon_-^{\rm f,b})^2 > (\delta_\beta Q^2)/(4g_1)$ is satisfied; this condition is reduced to the previous one by the corresponding renormalisation of coordinates ($\eta'=(g_1/\delta_\beta)^{1/2}\eta$ and $\zeta'=g_1\zeta$) in Eqn (4) or (5).

In all cases implying propagation of a single mode, the dispersion relations between the amplitude perturbation growth increment ImK and the transverse wavenumber Q are similar to those considered in [10–38]. This increment is nonzero in the range $0 < |Q| < 2\Upsilon_+^{\rm f,b}$ and reaches the maximum value ${\rm Im}K_{\rm max} = (\Upsilon_+^{\rm f,b})^2$ at $Q = \sqrt{2}\,\Upsilon_+^{\rm f,b}$. In dimensional units the maximum perturbation growth increment has an order of the coupling constant $|\gamma|$, which amounts to several tens of inverse centimetres for the parameters accepted above. This value is in agreement with the small-scale self-focusing length $1/|\gamma| \approx 10^{-2}$ cm, which was used when discussing the results of the analysis of the modulation instability of guided modes of a thin right-handed film on a nonlinear substrate, which was performed in [20]. Note that the exact analytical description of the nonlinear evolution of modulation instability of a plane wave in right-handed media [11, 35] demonstrates periodicity of instability development along the longitudinal coordinate.

5. Modulation instability for two counterpropagating modes with positive or negative group velocities

There are two possible versions for this case: (i) two counterpropagating modes with a positive group velocity ($\Upsilon_+^{\rm f} \neq 0$ and $\Upsilon_+^{\rm b} \neq 0$ at $\Upsilon_-^{\rm f} = \Upsilon_-^{\rm b} = 0$) and (ii) two counterpropagating modes with a negative group velocity ($\Upsilon_-^{\rm f} \neq 0$ and $\Upsilon_-^{\rm b} \neq 0$ at $\Upsilon_+^{\rm f} = \Upsilon_+^{\rm b} = 0$).

According to methodology applied in the linear analysis of stability [10–62], we will consider the additive perturbations of the amplitude coefficients of waveguide modes $\Upsilon_{\rm down}^{\rm up}$ [the indices up and down are, respectively, the superscript and subscript in (9), corresponding to the case under consideration] in the form

$$u_{\rm down}^{\rm up}(\eta,\zeta) = a_{\rm down}^{\rm up}\exp[{\rm i}(K\zeta+Q\eta)]$$

$$+\,b_{\rm down}^{\rm up}\exp[-{\rm i}(K\zeta+Q\eta)]. \qquad (14)$$

After the linearization (with respect to the perturbation amplitudes $a_{\rm down}^{\rm up}$ and $b_{\rm down}^{\rm up}$) of Eqns (2) and (3) [(4) and (5)], which describe the propagation of modes with only positive (negative) group velocity in the first (second) version under the conditions $C_-^{\rm f} \equiv C_-^{\rm b} \equiv 0$ ($C_+^{\rm f} \equiv C_+^{\rm b} \equiv 0$), one can easily derive a biquadratic equation for the longitudinal wavenumber K. Its solution can be written as

$$K_{1,2}^2\frac{2}{Q^2} = \frac{Q^2}{2} \mp {\rm sign}\,n_{2\rm s}\left[(\Upsilon_\pm^{\rm f})^2+(\Upsilon_\pm^{\rm b})^2\right]$$

$$\mp\sqrt{(\Upsilon_\pm^{\rm f})^4+14(\Upsilon_\pm^{\rm f}\Upsilon_\pm^{\rm b})^2+(\Upsilon_\pm^{\rm b})^4}, \qquad (15)$$

where the upper and lower signs before the function sign$n_{2\rm s}$ correspond to the first and second versions, respectively. In both versions, the upper and lower signs before the square root correspond to the first (K_1^2) and second (K_2^2) roots of biquadratic equation, respectively.

The parameters δ_β and g_1, which are present in the simplified (for the second version) NSEs (4) and (5), are absent in the latter formula because of the renormalisation of coordinates ($\eta'=(g_1/\delta_\beta)^{1/2}\eta$ and $\zeta'=g_1\zeta$), which in no way affects the presence or absence of modulation instability in this mode propagation version.

It follows from relation (15) that the modulation instability, to which complex values of perturbation longitudinal wavenumber K correspond, is possible for both versions at any sign of coefficient $n_{2\rm s}$, if the following condition is fulfilled for the amplitude coefficients:

$$\left\{\left[(\Upsilon_\pm^{\rm f})^2+(\Upsilon_\pm^{\rm b})^2\right]^2+12(\Upsilon_\pm^{\rm f}\Upsilon_\pm^{\rm b})^2\right\}^{1/2}$$

$$\pm\,{\rm sign}\,n_{2s}\left[(\Upsilon_\pm^{\rm f})^2+(\Upsilon_\pm^{\rm b})^2\right] > \frac{Q^2}{2}. \qquad (16)$$

This condition relates the mode intensities (proportional to squared amplitude coefficients) to the transverse wavenumber of amplitude perturbations, in the same way as in the case of propagation of a single mode in the film.

The dispersion relations between the normalised perturbation growth increment $\kappa = |\mathrm{Im}(K/\Upsilon_+^{\mathrm{f}})|$ and the normalised transverse perturbation wavenumber $q = Q/\Upsilon_+^{\mathrm{f}}$, , which characterise the presence of modulation instability of two counterpropagating modes of thin left-handed film with positive group velocities at positive ($n_{2\mathrm{s}} > 0$) and negative ($n_{2\mathrm{s}} < 0$) Kerr coefficients of the substrate, are shown in Fig. 2 by solid and dashed lines, respectively, for different ratios of mode amplitude coefficients δ_Υ. In this case, the instability growth increment is larger for the positive coefficient $n_{2\mathrm{s}}$. The increment for $n_{2\mathrm{s}} > 0$ and $\delta_\Upsilon = 0$ is equal to the increment for $n_{2\mathrm{s}} < 0$ and $\delta_\Upsilon = 1$. Obviously, in the case of two counterpropagating modes with negative group velocities, the solid (dashed) lines correspond, vice versa, to the negative (positive) Kerr coefficient. Thus, the conditions for development of modulation instability of two modes with a negative group velocity are inverted with respect to the conditions for the instability development in the case of two modes having a positive group velocity.

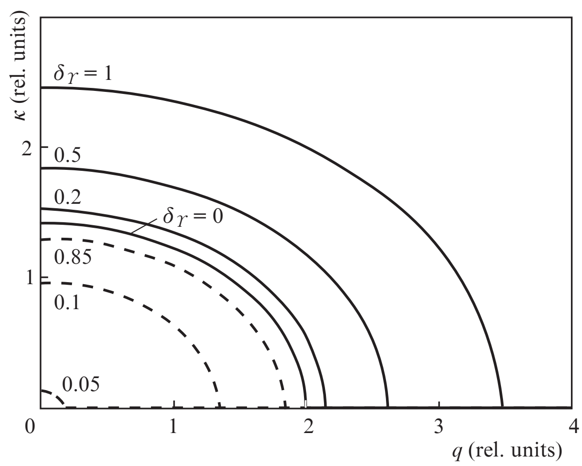

Figure 2. Dispersion relations for the growth increment $\kappa(q)$ of modulation instability of two counterpropagating modes of a thin left-handed film with positive group velocities for positive ($n_{2\mathrm{s}} > 0$, solid line) and negative ($n_{2\mathrm{s}} < 0$, dashed line) Kerr coefficients of the substrate at different ratios of mode amplitude coefficients δ_Υ. The curve for $n_{2\mathrm{s}} > 0$ and $\delta_\Upsilon = 0$ merges with the curve for $n_{2\mathrm{s}} < 0$ and $\delta_\Upsilon = 1$.

It follows from Fig. 2 and formula (16) that, both in the first and second versions, the condition for modulation instability is fulfilled at a lower total mode intensity for that sign of substrate Kerr coefficient at which it can be was fulfilled for only one mode propagating in the waveguide.

6. Modulation instability in the case of copropagating modes with opposite signs of group velocities

This case corresponds to forward or backward propagating modes, one of which has a positive group velocity and the other has a negative group velocity, e.g., to forward propagating modes with opposite signs of group velocities ($\Upsilon_+^{\mathrm{f}} \neq 0$, $\Upsilon_-^{\mathrm{f}} \neq 0$, and $\Upsilon_+^{\mathrm{b}} = \Upsilon_-^{\mathrm{b}} = 0$). Similar to the previous case (Section 5), the relation for the squared longitudinal wavenumber of mode amplitude perturbations [see (15)] can easily be derived from NSEs (2) and (4) under the conditions $C_+^{\mathrm{b}} \equiv C_-^{\mathrm{b}} \equiv 0$, after their linearization with respect to the perturbation amplitudes $a_{\mathrm{down}}^{\mathrm{up}}$ and $b_{\mathrm{down}}^{\mathrm{up}}$ in the following form:

$$K_{1,2}^2\frac{2}{Q^2} = \frac{1+\delta_\beta^2}{4}Q^2 + \mathrm{sign}n_{2\mathrm{s}}\left[\delta_\beta g_1(\Upsilon_-^{\mathrm{f}})^2 - (\Upsilon_+^{\mathrm{f}})^2\right] \pm\left\{\frac{(1-\delta_\beta^2)^2}{16}Q^4 - \mathrm{sign}n_{2\mathrm{s}}\frac{1-\delta_\beta^2}{16}\left[(\Upsilon_+^{\mathrm{f}})^2 + \delta_\beta g_1(\Upsilon_-^{\mathrm{f}})^2\right] + (\Upsilon_+^{\mathrm{f}})^4 - 2\delta_\beta(8g^2 - g_1)(\Upsilon_+^{\mathrm{f}}\Upsilon_-^{\mathrm{f}})^2 + \delta_\beta^2 g_1^2(\Upsilon_-^{\mathrm{f}})^4\right\}^{1/2}, \quad (17)$$

where the upper and lower signs before the square root correspond, respectively, to the first and second roots of the biquadratic equation.

As follows from the latter relation, under the conditions accepted, the existence of modulation instability is determined by not only the mode intensities and transverse wavenumber Q but also by the ratio of the waveguide mode propagation constants δ_β, as well as the parameters g and g_1, which describe the influence of the difference in the spatial distributions of modes on their overlap in a nonlinear substrate [compare with (15)].

Near the frequency corresponding to zero group velocity, the values δ_β, g, and g_1 are close to unity ($g_1 < g < \delta_\beta < 1$) [71, 82]. It follows from formula (15) that a small deviation of the coefficients g and g_1 from unity causes only weakly pronounced quantitative changes in the dependences of longitudinal wavenumber K on the transverse wavenumber Q. At the same time, even a small deviation of the ratio of waveguide mode propagation constants, δ_β, from unity may change significantly the character of these dependences. Note that, at $\delta_\beta = g_1 = g = 1$, the radicand in formula (17) may take negative values, in contrast to the radicand in formula (15), which is always positive.

If we neglect the difference in the mode propagation constants, the following strict equality holds true: $\delta_\beta = 1$. In this case, the wavenumber K is complex for any real Q and any sign of substrate Kerr coefficient $n_{2\mathrm{s}}$, if the following condition is satisfied:

$$2 - \sqrt{3} \approx (2g/g_1) - \sqrt{(2g/g_1)^2 - 1} < \delta_\Upsilon < (2g/g_1) + \sqrt{(2g/g_1)^2 - 1} \approx 2 + \sqrt{3} \quad (18)$$

(this condition is imposed on the ratio of amplitude coefficients: $\delta_\Upsilon = \Upsilon_-^{\mathrm{f}}/\Upsilon_+^{\mathrm{f}}$). The dependences of the normalised perturbation growth increment κ on the normalised transverse perturbation wavenumber q, characterising the presence of modulation instability for $n_{2\mathrm{s}} > 0$ and $g_1 = g = 1$, are shown in Fig. 3a. Since the radicand in formula (17) at $\delta_\beta = g_1 = g = 1$ may take negative values independent of the transverse wavenumber, no threshold condition [similar to inequality (16)] is imposed on the mode intensities and transverse wavenumber in this case. The increment κ increases with a rise in the ratio δ_Υ in the range under consideration.

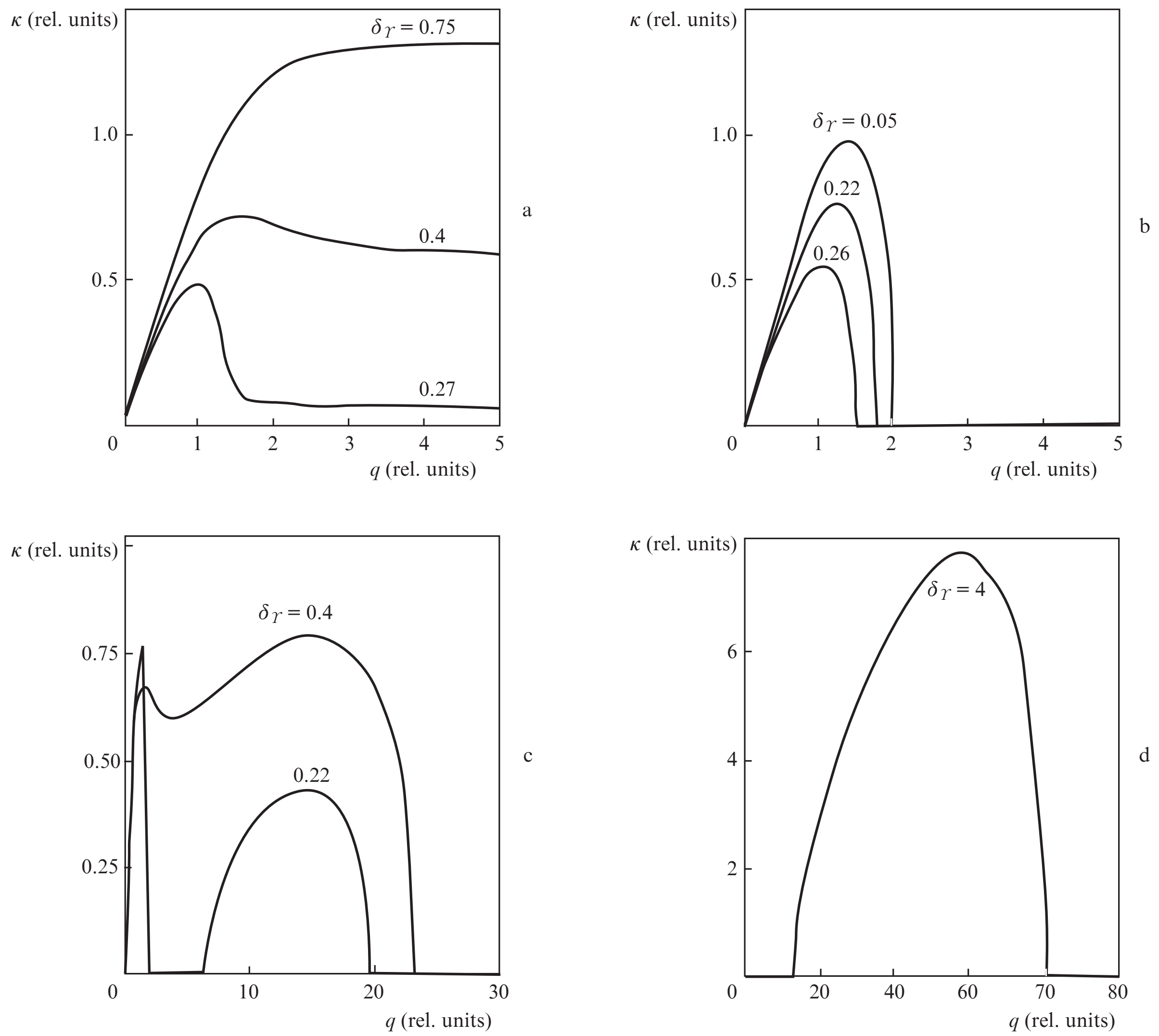

Figure 3. Dispersion relations $\kappa(q)$, characterising the modulation instability of two forward propagating modes of a thin left-handed film, in the case of opposite signs of mode group velocities and substrate positive Kerr coefficient for (a, b) $\delta_\beta = g_1 = g = 1$ and (c, d) $\delta_\beta = 0.99$, $g_1 = 0.98$, and $g = 0.97$ at different δ_Υ.

If the intensity ratio δ_Υ lies in the range $0 < \delta_\Upsilon < (2g/g_1) - \sqrt{(2g/g_1)^2 - 1} \approx 2 - \sqrt{3}$, at $\delta_\beta = 1$, the modulation instability is also implemented if the threshold condition, which can be written for both signs of the coefficient n_{2s} as

$$\left[(\Upsilon_+^{\rm f})^4 - 2(8g^2 - g_1)(\Upsilon_+^{\rm f}\Upsilon_-^{\rm f})^2 + g_1^2(\Upsilon_-^{\rm f})^2\right]^{1/2}$$

$$+\,{\rm sign}n_{2s}\left[(\Upsilon_+^{\rm f})^2 - (\Upsilon_-^{\rm f})^2\right] > \frac{Q^2}{2} \qquad (19)$$

is fulfilled. Figure 3b presents the dispersion relations $\kappa(q)$ for the δ_Υ values from the latter range. It is noteworthy that in this case, in contrast to the situation corresponding to Fig. 3a, the increment κ, vice versa, decreases with an increase in the ratio δ_Υ. If δ_Υ satisfies the inequality $\delta_\Upsilon > (2g/g_1) + \sqrt{(2g/g_1)^2 - 1} \approx 2 + \sqrt{3}$, the modes propagating in a film with oppositely directed group velocities are not subjected to modulation instability.

Figures 3c and 3d show the dependences $\kappa(q)$ calculated for the parameters $g_1 = g = 1$ (neglecting the difference in the mode spatial distributions) and the ratio $\delta_\beta = 0.99$, taking into account the small difference in the propagation constants of forward propagating modes with oppositely directed group velocities at a frequency near zero group velocity [69–71, 82]. A comparison of the curves in Figs 3a and 3c at the same value $\delta_\Upsilon = 0.4$ shows that, at a deviation of the ratio δ_β from unity, the range of transverse wavenumbers at which modulation instability may develop becomes finite. Another essential change in the dependences $\kappa(q)$, related to the deviation of ratio δ_β from unity, follows from a comparison of the curves in Figs 3b and 3c at $\delta_\Upsilon = 0.22$. The consequence of this deviation is as follows: with an increase in the transverse wavenumber, there arises another region in which modulation instability develops. One more essential feature caused by the deviation of δ_β from unity is the presence of a region of modulation instability in Fig. 3d at the amplitude coefficient ratio $\delta_\Upsilon = 4$, for which, as was indicated above, modulation instability is absent when the strict equality $\delta_\beta = 1$ is fulfilled.

Note that the presence of modulation instability in the case of two counterpropagating modes, one of which has a positive group velocity and the other has a negative group velocity, follows from NSEs (2) and (5) under the conditions $C_+^{\mathrm{b}} \equiv C_-^{\mathrm{f}} \equiv 0$. In this case the conditions for the modulation instability development are completely similar to those considered in this section.

7. Conclusions

Thus, the existence of modulation instability of fast TE modes of a thin left-handed film on a right-handed substrate exhibiting the Kerr effect at a frequency near zero of their group velocity can be established by analysing the solutions to four coupled NSEs. A solution to these equations, similar to the plane-wave solution of the nonlinear wave equation for a bulk medium, can be obtained in the general case of two forward propagating modes with positive and negative group velocities and two backward propagating modes, also with positive and negative group velocities. The simultaneous propagation of all four waveguide modes is not only accompanied by perturbation of their propagation constants, which depend on the mode intensities; it also calls for matching the mode amplitude coefficients. This requirement is directly related to the contribution of spatial parametric mixing, which occurs under conditions of simultaneous propagation of all four modes. The aforementioned matching is absent in the case of propagation of any three or two modes or a single mode.

When a single mode propagates, its modulation instability may occur both at a positive Kerr coefficient of the substrate, if the mode has a positive group velocity, and at a negative Kerr coefficient, if the mode group velocity is negative. In both cases the conditions for developing modulation instability, determined by the existence of imaginary part for the longitudinal wavenumber of mode amplitude perturbations, are identical; they relate the mode intensity to the transverse wavenumber of these perturbations. At a specified intensity of a mode, its modulation instability is possible in a limited range of transverse wavenumbers.

At a frequency near zero mode group velocity, there are two versions of counterpropagating modes with the same signs of group velocities. One corresponds to the propagation of modes with positive group velocities, and the other corresponds to the propagation of modes with negative group velocities. In each version modulation instability may develop at any sign of substrate Kerr coefficient. The conditions for its development, which relate the mode intensity to the transverse wavenumber of amplitude perturbations, can be considered as inverted with respect to each other in their qualitative character. Note that in the case of the first (second) version, the development of modulation instability for a positive (negative) Kerr coefficient is possible at a lower total intensity of waveguide modes than for a negative (positive) coefficient.

In the case of copropagating or counterpropagating modes with opposite signs of group velocities, the conditions for developing modulation instability depend not only on the mode intensities and the transverse wavenumber of perturbations but also on the difference in the propagation waveguide mode constants. This difference affects significantly the dispersion relations between the longitudinal and transverse wavenumbers and may change their qualitative character. Modulation instability may develop at any sign of substrate Kerr coefficient for perturbation transverse wavenumber, either from one continuous region or from two separated regions, depending on the ratio of mode amplitude coefficients.

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References

1. Kivshar Yu. *Natl. Sci. Rev.*, **5**, 144 (2018).
2. Li G., Zhang S., Zentgraf T. *Nat. Rev. Mater.*, **2**, 14 (2017).
3. Glybovski S.B., Tretyakov S.A., Belov P.A., Kivshar Yu.S., Simovski C.R. *Phys. Rep.*, **634**, 1 (2016).
4. Lapine M., Shadrivov I., Kivshar Yu. *Rev. Mod. Phys.*, **86**, 1093 (2014).
5. Bespalov V.I., Talanov V.I. *JETP Lett.*, **3**, 307 (1966) [*Pis'ma Zh. Eksp. Teor. Fiz.*, **3**, 471 (1966)].
6. Ostrovskii L.A. *J. Exp. Theor. Phys.*, **24**, 797 (1967) [*Zh. Eksp. Teor. Fiz.*, **51**, 1189 (1966)].
7. Karpman V.I. *JETP Lett.*, **6**, 277 (1967) [*Pis'ma Zh. Eksp. Teor. Fiz.*, **6**, 829 (1967)].
8. Gurovich V.Ts., Karpman V.I. *J. Exp. Theor. Phys.*, **29**, 1048 (1969) [*Zh. Eksp. Teor. Fiz.*, **56**, 1952 (1969)].
9. Kolokolov A.A. *Lettere AL Nuovo Cimento*, **8**, 197 (1973).
10. Rosanov N.N. *Progress in Optics*, **35**, 1 (1996).
11. Akhmediev N.N., Ankevich A. *Nonlinear Pulses and Beams* (London, Weinhem, New York, Tokyo, Melbourne, Madras: Chapman & Hall, 1997; Moscow: Fizmatlit, 2003).
12. Kivshar Yu.S., Agraval G.P. *Optical Solitons: From Fibers to Photonic Crystals* (San Diego: Acad. Press, 2003; Moscow: Fizmatlit, 2005).
13. Agrawal G.P. *Nonlinear Fiber Optics* (Boston: Academic, 1989; Moscow: Mir, 1996).
14. Rozanov N.N., Smirnov V.A. *J. Exp. Theor. Phys.*, **43**, 1075 (1976) [*Zh. Eksp. Teor. Fiz.*, **70**, 2060 (1976)].
15. Rozanov N.N., Smirnov V.A. *Sov. J. Quantum Electron.*, **8**, 1429 (1978) [*Kvantovaya Elektron.*, **5**, 2538 (1978)].
16. Rozanov N.N., Smirnov V.A. *Sov. J. Quantum Electron.*, **20**, 1250 (1990) [*Kvantovaya Elektron.*, **17**, 1340 (1990)].
17. Rozanov N.N., Smirnov V.A. *Sov. J. Quantum Electron.*, **10**, 232 (1980) [*Kvantovaya Elektron.*, **7**, 410 (1980)].
18. Rozanov N.N. *Opt. Spektrosk.*, **48**, 338 (1989).
19. Vysotina N.V., Rozanov N.N., Smirnov V.A. *Zh. Tekh. Fiz.*, **57**, 173 (1987).
20. Vyssotina N.V., Rosanov N.N., SmirnovV.A. *J. Opt. Soc. Am. B*, **7**, 1281 (1990).
21. Hasegawa A., Brinkman W.F. *IEEE J. Quantum Electron.*, **16**, 694 (1980).
22. Hasegawa A. *Opt. Lett.*, **9**, 288 (1984).
23. Anderson D., Lisak M. *Opt. Lett.*, **9**, 468 (1984).
24. Hermansson B., Yevick D. *Opt. Commun.*, **52**, 99 (1984).
25. Tajima K. *J. Lightwave Technol.*, **4**, 900 (1986).
26. Hasegawa A., Tomita A. *Phys. Rev. Lett.*, **56**, 135 (1986).
27. Tai K., Tomita A., Jewell J.L., Hasegawa A. *Appl. Phys. Lett.*, **49**, 236 (1986).
28. Shukla P.K., Rasmussen J.J. *Opt. Lett.*, **11**, 171 (1986).
29. Kothari N.C. *Opt. Commun.*, **36**, 247 (1987).
30. Potasek M.J., Agrawal G.P. *Phys. Rev. A*, **56**, 3862 (1987).
31. Jones C.K.R.T., Moloney J.V. *Phys. Lett. A*, **117**, 175 (1986).
32. Moloney J.V., Ariyasu J., Seaton C.T., Stegeman G.I. *Appl. Phys. Lett.*, **48**, 826 (1986).
33. Ariyasu J., Seaton C.T., Stegeman G.I. *IEEE J. Quantum Electron.*, **22**, 984 (1986).
34. Akhmediev N.N., Ostrovskaya N.V. *Zh. Tekh. Fiz.*, **58**, 2194 (1988).
35. Akhmediev N.N., Korneev V.I., Mitskevich N.V. *J. Exp. Theor. Phys.*, **77**, 89 (1988) [*Zh. Eksp. Teor. Fiz.*, **94**, 159 (1988)].
36. Leine L., Wachter C., Langbein U., Lederer F. *J. Opt. Soc. Am. B*, **5**, 547 (1988).
37. Tran H.T., Ankiewicz A. *IEEE J. Quantum Electron.*, **28**, 488 (1992).
38. Malendevich R., Jankovic L., Stegeman G., Aitchison J.S. *Opt. Lett.*, **26**, 1879 (2001).
39. Agrawal G.P. *Phys. Rev. Lett.*, **59**, 880 (1987).
40. Rothenberg J.E. *Phys. Rev. A*, **42**, 682 (1990).
41. Aceves A.B., De Angelis C., Wabnitz S. *Opt. Lett.*, **17**, 1758 (1992).

42. Haelterman M., Sheppard A.P. *Phys. Rev. E*, **49**, 3389 (1994).
43. Tran H.T., Sammut R.A. *Opt. Commun.*, **119**, 583 (1995).
44. Seve E., Dinda P.T., Millot G., Remoissenet M., Bilbault J.M., Haelterman M. *Phys. Rev. A*, **54**, 3519 (1996).
45. Seve E., Millot G., Trillo S. *Phys. Rev. E*, **61**, 3139 (2000).
46. Millot G., Pitois S., Dinda P.T. *J. Opt. Soc. Am. B*, **19**, 454 (2002).
47. Malendevich R.R., Friedrich L., Stegeman G.I., Soto-Crespo J.M., Akhmediev N.N., Aitchison J.S. *Opt. Soc. Am. B*, **19**, 695 (2002).
48. Chiu H.S., Chow K.W. *Phys. Rev. A*, **79**, 065803 (2009).
49. Kourakis I., Shukla P.K. *Phys. Rev. E*, **72**, 016626 (2005).
50. Wen S., Xiang Y., Su W., Hu Y., Fu X., Fan D. *Opt. Express*, **14**, 1568 (2006).
51. Wen S., Wang Y., Su W., Xiang Y., Fu X., Fan D. *Phys. Rev. E*, **73**, 036617 (2006).
52. Xiang Y., Wen S., Dai X., Tang Z., Su W., Fan D. *J. Opt. Soc. Am. B*, **24**, 3058 (2007).
53. Wen S., Xiang Y., Dai X., Tang Z., Su W., Fan D. *Phys. Rev. A*, **75**, 033815 (2007).
54. Dai X., Xiang Y., Wen S., Fan D. *J. Opt. Soc. Am. B*, **26**, 564 (2009).
55. Sarma A.K., Saha M. *J. Opt. Soc. Am. B*, **28**, 944 (2011).
56. Xiang Y., Wen S., Dai X., Fan D. *J. Opt. Soc. Am. B*, **28**, 908 (2011).
57. Zhong X., Tang T., Xiang A., Cheng K. *Opt. Commun.*, **284**, 4727 (2011).
58. Xiang Y., Dai X., Wen S., Fan D. *Phys. Rev. E*, **82**, 056605 (2010).
59. Zhang L., Xiang Y., Dai X., Wen S. *J. Opt. Soc. Am. B*, **31**, 3029 (2014).
60. Inc M., Aliyua A.I., Yusufa A., Baleanuc D. *Optik*, **157**, 484 (2018).
61. Zanga D., Fewo S.I., Tabi C.B., Kofanéa T.C. *Commun. Nonlinear Sci. Numer. Simul.*, **80**, 104993 (2020).
62. Zhang L., Wang Q. *Optik*, **227**, 166090 (2021).
63. Novitsky A.V., Barkovsky L.M. *J. Opt. A: Pure Appl. Opt.*, **7**, S51 (2005).
64. Wang Z.J., Dong J.F. *Prog. Electromagn. Res.*, **62**, 203 (2006).
65. Wang Z.H., Xiao Z.Y., Li S.P. *Opt. Commun.*, **281**, 607 (2008).
66. Wang Z.H., Xiao Z.Y., Luo W.Y. *J. Opt. A: Pure Appl. Opt.*, **11**, 015101 (2009).
67. Shen L.F., Wang Z.H. *J. Opt. Soc. Am. A*, **26**, 754 (2009).
68. Taya S.A., Kullab H.M., Qadoura I.M. *J. Opt. Soc. Am. B*, **30**, 2008 (2013).
69. Konkin D.A., Shibel'gut A.A., Litvinov R.V. *Izv. Samar. Nauch. Tsentra RAN*, **17**, 83 (2015)].
70. Konkin D.A., Litvinov R.V., Parfenova E.S., Rakhim R.A.A., Stukach O.V. *Quantum Electron.*, **46**, 1040 (2016) [*Kvantovaya Elektron.*, **46**, 1040 (2016)].
71. Buller A.S., Leonov S.V., Litvinova N.R., Litvinov R.V. *J. Exp. Theor. Phys.*, **130**, 387 (2020) [*Zh. Eksp. Teor. Fiz.*, **157**, 461 (2020)].
72. Barnoski M. (Ed.) *Introduction to Integrated Optics* (New York: Plenum, 1974; Moscow: Mir, 1977).
73. Tamir T. (Ed.) *Integrated Optics* (New York: Springer, 1975; Moscow: Mir, 1978).
74. Hunsperger R.G. *Integrated Optics: Theory and Technology* (Berlin: Springer, 1982; Moscow: Mir, 1985).
75. Unger H.G. *Planar Optical Waveguides and Fibres* (Oxford: Clarendon, 1977; Moscow: Mir, 1977).
76. Snyder A.W., Love J.D. *Optical Waveguide Theory* (New York: Chapman and Hall, 1983; Moscow: Radio i Svyaz', 1987).
77. Stegeman G.I., Wright E.M., Finlayson N., Zanoni R., Seaton C.T. *J. Lightwave Technol.*, **6**, 953 (1988).
78. Stegeman G.I., Seaton C.T. *J. Appl. Phys.*, **58**, R57 (1985).
79. Crosignani B., Cutolo A., Porto P.D. *J. Opt. Soc. Am.*, **72**, 1136 (1982).
80. Hasegawa A. *Opt. Lett.*, **5**, 416 (1980).
81. Crosignani B., Porto P.D. *Opt. Lett.*, **6**, 329 (1981).
82. Litvinov M.R., Spiridonova A.S., Litvinov R.V., Konkin D.A. *Pis'ma Zh. Tekh. Fiz.*, **47**, 48 (2021).