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Modulation and amplification of wave packets in amplifiers with a travelling refractive-index wave

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Abstract. **We study the conditions of frequency modulation, spectral broadening, and amplification of a Gaussian pulse – a whispering-gallery-mode (WGM) wave packet (WP) – propagating along a spiral trajectory on the surface of an active cylindrical optical waveguide, in which a travelling refractive-index wave (TRIW) is generated. Analytical expressions are obtained for the dependences of the duration, chirp, and spectral width on the distance travelled by the pulse along the waveguide, as well as on the parameters of the waveguide and the radiation launched into it. It is shown that the interaction of a wave packet with the TRIW leads to a strong frequency modulation of the amplified pulse with the conserved linearity of the chirp. It is demonstrated that this circumstance can be used to generate pico- and subpicosecond pulses with peak powers above 100 kW.**

Keywords: *optical waveguide, travelling refractive-index wave, pulse amplification, frequency modulation of pulses.*

1. Introduction

It is known that when a light pulse propagates through an optical waveguide with a travelling refractive-index wave (TRIW), one can observe effects that are absent in optical fibres with stationary parameters [1, 2]. For example, the authors of Refs [3–5] studied the effects associated with a change in polarisation and a shift of the carrier frequency of quasi-monochromatic wave packets under the influence of TRIW. The authors of papers [6–9] considered the formation of soliton-like wave packets, as well as the development of induced modulation instability and generation of picosecond pulses in such fibres. In this case, the synchronisation of the speeds of the TRIW and the modulated electromagnetic radiation is a challenging task. In this paper, we consider a synchronisation scheme for the TRIW and whispering-gallery-mode (WGM) wave packets propagating along the surfaces of the corresponding modulated waveguides.

Surface waves of the WGM type arise in axisymmetric systems and are formed at curved interfaces between two media and, despite their long history [10], remain interesting for researchers in various fields of physics. Electromagnetic WGMs can be detected, for example, in dielectric waveguides of spherical, spheroidal, toroidal, and cylindrical shapes [2, 11, 12]. In the latter case, wave packets of this kind propagate on the surface of a cylindrical silica waveguide along a spiral trajectory with a constant pitch, which is why they are called tunnelling modes [13]. Among the features of such spiral waves, it is important that their longitudinal (along the waveguide axis) group velocity can be arbitrarily less than the speed of light in vacuum [7, 14].

Let us consider the dynamics of a WGM-type wave packet in an optical waveguide with a TRIW. Modulated and amplified wave packets (WPs) are launched into a cylindrical waveguide in such a way that they propagate along a spiral trajectory (Fig. 1), with the pitch of the spiral determining the longitudinal component of the WP group velocity. Due to the small pitch of the spiral [7–9], it is possible to synchronise the speeds of the modulated WP and TRIW and, finally, to provide effective WP amplification and modulation.

2. Basic equations

When light is launched into a cylindrical waveguide at a certain angle θ to the generatrix of the cylinder (Fig. 2), the surface wave propagates along a spiral (see Fig. 1) trajectory [7]. The longitudinal (β_z) and transverse (radial, β_r) components of the wave vector $\beta = n_0\omega/c$ of such a wave are related by the expression $\beta_z = (\beta^2 - \beta_r^2)^{1/2}$, where $n_0(\omega)$ is the refractive index of the waveguide material. If the angle θ at which the wave is launched into the waveguide is small, i.e., the launching direction is close enough to the cross section of the waveguide (line *6* in Fig. 2), then the wave propagation along its axis is significantly slowed down to zero values (at $\theta = 0$) of the longitudinal velocity components: $V_z \to 0$.

In what follows, we consider the case of a surface wave slowly tunnelling along the longitudinal axis of the waveguide z with a velocity $V_z \ll c$, which is possible for $\beta_z \ll \beta \approx \beta_r$ In this case, the electric field of the wave can be represented as follows:

$$E(z,t,r,\varphi) = A(z,t)\Phi(z,r,\varphi) \times \exp\left(\mathrm{i}\omega t - \mathrm{i}\int_0^z \beta_z(z)\,\mathrm{d}z\right) + \mathrm{c.c.}, \quad (1)$$

where $A(z,t)$ is a slowly varying amplitude describing the longitudinal (along the z axis) distribution of the tunnelling wave field, and $\Phi(z,r,\varphi)$ is the function that determines the radial and azimuthal dependences of the field in the waveguide.

Suppose that aTRIW is excited in a cylindrical waveguide, and its refractive index (RI) is determined as [1–3, 7–9]

$$n(t,z) = n_0[1 - m\cos(\Omega t - qz + \psi)] + n^{(2)}I, \quad (2)$$

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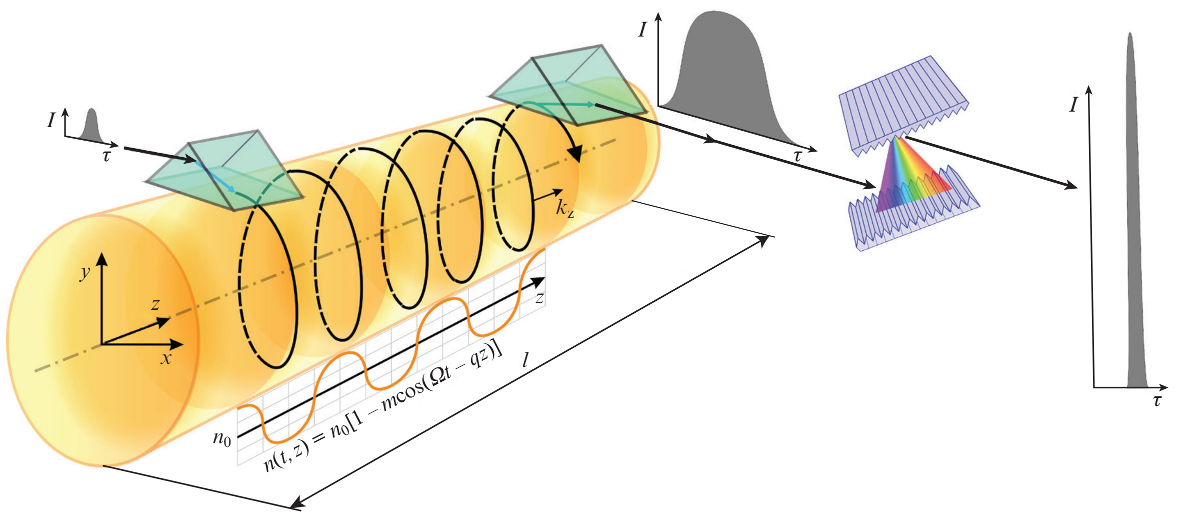

Figure 1. Schematic of the experiment on the amplification and frequency modulation of the WP in a waveguide with a TRIW and further compression of the packet. Shown is a prismatic version of radiation coupling into a waveguide, a tunnelling optical wave (spiral), a TRIW, input and output pulses, and a pair of diffraction grating compressors.

where Ω is the modulation frequency of the TRIW; $q = 2\pi/\Lambda$ is the wave number; Λ is the period of the TRIW spatial inhomogeneity; $V_m = \Omega/q$ is the speed of TRIW movement (phase speed); $m = \Delta n/n_0$ is the TRIW modulation depth; Δn is the amplitude of the TRIW change; ψ is the phase shift, which is determined by the time of the mismatch at the launch point between the TRIW maximum and the WP envelope maximum; $I = |A|^2$ is the intensity of propagating radiation; and $n^{(2)}$ is the nonlinear refractive index [15].

Let a Gaussian frequency-modulated (FM) pulse of the form [15, 16]

$$A(z = 0, t) = A_0\exp[-(\tau_0^{-2}/2 - i\alpha_0)t^2], \tag{3}$$

be launched, along with the TRIW of form (2), into the waveguide at an angle θ to its cross section, where A_0 is the peak value of the pulse amplitude at the entrance to the waveguide; and τ_0 and α_0 are the initial duration and rate of frequency modulation (chirp) of the pulse. In this case, the trajectory of the wave packet has the form of a spiral, the pitch of which depends on the angle θ (Fig. 2). In synchronising the speeds of the WP and TRIW, when the condition $V_z = V_m$ is satisfied, we have $\sin\theta = nV_z/c = nV_m/c \approx \theta$. In addition, we assume that the WP and TRIW are synchronised in time, when the maximum of the WP envelope moves together with the TRIW maximum, i.e., $\psi = 0$.

Let us go over to a frame of reference associated with a spiralling optical pulse. The effective optical length ξ travelled by the pulse over the waveguide surface is related to the angle of its launching θ: $\partial\xi = \partial z/\sin\theta \approx (c/nV_m)\partial z$, and the running time associated with the WP is $\tau = t - (\partial\beta/\partial\omega)_{\omega=\omega_0}\xi \approx t - n\xi/c$. Then the WP dynamics in the coordinates (ξ, τ), taking into account (2), can be described by the relation [15]:

$$\frac{\partial A}{\partial \xi} - i\frac{d_2}{2}\frac{\partial^2 A}{\partial \tau^2} + \frac{d_3}{6}\frac{\partial^3 A}{\partial \tau^3} + iR\left(|A|^2 - \tau_R\frac{\partial |A|^2}{\partial \tau}\right)A$$

$$= i\beta m\cos[\Omega(\tau - \delta\tau)]A + gA. \tag{4}$$

Here $d_n = (\partial^n\beta/\partial\omega^n)_{\omega_0}$ $(n = 1, 2, 3)$ are higher-order dispersion parameters; τ_R is a parameter characterising the effect of stimulated Raman self-scattering of a medium [15, 16], approximately equal to the nonlinear time of its response; $g(\omega) = g_0[1 + (\omega - \omega_{res})^2/\Delta\omega_{lin}^2]^{-1}$ is the gain of an active waveguide; ω_{res} is the resonant frequency of the gain line; $\Delta\omega_{lin}$ is the width of the gain line of the active medium; and $R = \omega_0 n^{(2)}/(cS_{eff})$ is the nonlinearity parameter, where

$$S_{eff}(z) = \left(\int_{-\pi}^{\pi}\int_0^{\infty} r|\Phi(z,r,\varphi)|^2 d\varphi dr\right)^2$$

$$\times\left(\int_{-\pi}^{\pi}\int_0^{\infty} r|\Phi(z,r,\varphi)|^4 d\varphi dr\right)^{-1} \tag{5}$$

is the effective area of the wave packet mode.

It should be also noted that at a sufficiently large refractive-index (RI) modulation depth, when $|m| \gg 10^{-5}$, we can talk about the possibility of 'pulling' the wave packet into the region of the maximum refractive index of the TRIW even if initially the WGM-type WP does not have a longitudinal component of the group velocity, that is, when $\beta_z = 0$ and $\beta_r = \beta$.

Thus, two scenarios are described for the implementation of slow wave tunnelling along a cylinder with a TRIW propagating in it. Obviously, the optimal synchronisation of the tunnelling WP and the TRIW is achieved as a result of a combination of two factors: the launching of the WP at a small angle θ, at which $V_z \approx V_m$, and the maintenance of the autosynchronisation regime of the WP with the RI maximum due to the large modulation depth m. Moreover, both in the first and in the second case, we can assume with a good degree of accuracy that $\delta\tau \to 0$.

In practice, the considered interaction of the WP and TRIW can be realised when an acoustic wave with a phase velocity $V_m \approx 6000$ m s^{-1} is excited in the waveguide. In a silica waveguide with a standard RI value $n \approx 1.5$, the synchronisation of the tunnelling WP and TRIW is provided when $\theta \approx V_m n/c \approx 3 \times 10^{-5}$. In this case, the modulation depth m can reach large values: $m = 4 \times 10^{-4}$ [17].

Amplification in a corresponding cylindrical waveguide with a large surface area can be realised by doping it (for example, with erbium, bismuth, or ytterbium ions) and using standard methods of pumping through the cladding [18–20].

Note that the proposed scheme for generating broadband pulses has certain analogies with the travelling pump wave regime known in quantum electronics, in which the wave motion through the active medium is synchronised with the motion of the generated pulse [17, 21, 22]. Probably, the modulation depth in this case can reach (not exceed) $m \sim 10^{-5}$, and the effective interaction length can significantly exceed meter lengths.

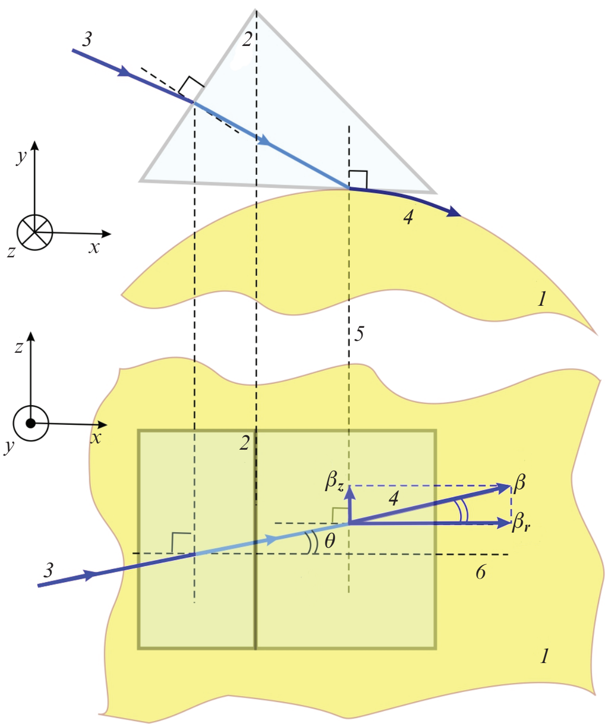

Figure 2. Scheme of radiation coupling into the optical waveguide using a prism [23]: a view of the prism and the optical waveguide from the side (top) and from above (below):
(*1*) waveguide; (*2*) prism; (*3*) launched radiation; (*4*) surface wave; (*5*, *6*) generatrix of a cylindrical waveguide and its cross section; θ is the angle between the direction of radiation coupling into the cylindrical waveguide and its cross section; β, β_z, and β_r are the wave vector and its longitudinal and transverse components.

A numerical analysis of Eqn (4), taking into account the influence of dispersion effects of higher orders, shows that in the section of the active waveguide, for which the inequality $\xi \ll \tau_0^2/|d_2|$, is valid, we can assume with a good degree of accuracy (see the Appendix) that

$$\alpha(\xi) \approx \alpha_0 + mn_0\omega_0\Omega^2\xi/c; \qquad (6a)$$

moreover, the WP duration at this length remains practically unchanged, i.e.

$$\tau_p(\xi) = \tau_0. \qquad (6b)$$

In this case, when the WP interacts with the TRIW, its ultrafast modulation is possible with the conserved linearity of the chirp and an increasing spectral width. This, in turn, makes it possible to further compress the pulse using standard methods, for example, on diffraction gratings (see Fig. 1). If, on the other hand, a negative chirp is formed in the pulse, then use can be made of a conventional normal-dispersion fibre to further compress the pulse.

In accordance with [15], the minimum possible pulse duration after its compression is

$$\tau_{\min} \approx \Delta\omega^{-1} \approx |\alpha(L)\tau_p(L)|^{-1} \qquad (7)$$

(L is the length of the path travelled by the WP over the waveguide surface), and the width of the pulse line with an envelope has the form (2)

$$\Delta\omega = \tau_p^{-1}\sqrt{1+\alpha^2\tau_p^4}. \qquad (8)$$

Let an initial pulse with a duration $\tau_0 = 10^{-11}$ s and a zero initial chirp $\alpha_0 = 0$ (i.e., the pulse is transform limited) is introduced into a waveguide modulator with the following parameters: modulation depth and frequency, $m\beta = \pm 10^4$ m^{-1} and $\Omega = 5 \times 10^{10}$ s^{-1}, respectively; and the dispersion of group velocities, $|d_2| = 10^{-26}$–10^{-27} s^2 m^{-1}. Then, at the exit from such a waveguide with a length of $l = 4$ cm, the pulse will acquire an effective chirp $|\alpha(L)| \approx 10^{24}$ s^{-2}, and its duration will not change significantly. Subsequently, after passing such a pulse through a dispersing element that provides temporary compression of the pulse (for example, through a diffraction grating, see Fig. 1), its duration can be reduced by a factor of 100, to 10^{-13} s.

3. Numerical simulation

With the help of numerical simulation of propagation equations (4), we will consider the possibilities of spectral broadening and amplification of an FM pulse of the form (2) while maintaining the linear velocity of frequency modulation in an active medium with a TRIW. In the numerical simulation, use was made of the split-step Fourier method (SSFM) [15].

Figure 3 shows the dynamics of the envelope of an FM pulse (Fig. 3a) and its spectrum (Fig. 3b) in an active waveguide with a realised TRIW with nonlinearity $R = 10^{-7}$, 10^{-5}, 10^{-3} W^{-1} m^{-1} [curves (*1*–*3*), respectively]. The initial parameters of the input pulse are as follows: duration $\tau_0 = 10^{-11}$ s, and peak power $P_0 = 10$ W. In addition, we assume that the pulse at the input to the waveguide modulator is transform limited, i.e., $\alpha_0 = 0$. The parameters of the medium with the realised TRIW are as follows: TRIW frequency $\Omega = 10^9$ s^{-1}, RI modulation depth $\Delta n = 10^{-4}$, wavenumber $\beta = 10^7$ m^{-1}, second-order normal variance $d_2 = 10^{-26}$ s^2 m^{-1}, and third-order variance $d_3 = 10^{-39}$ s^3 m^{-1}. The optical path length of the pulse in the waveguide, ξ, is taken equal to 500 m. In this case, it is assumed that $\delta\tau \to 0$. The parameters of the gain line of the medium are as follows: $g(\omega) = g_0[1 + (\omega - \omega_0)^2/\Delta\omega_{\rm lin}^2]^{-1}$, $g_0 \approx 10^{-2}$ m^{-1}, and $\Delta\omega_{\rm lin} = 10^{11}$ s^{-1}. As can be seen from Fig. 3b, the pulse acquires an almost linear chirp, broadens significantly, and its peak power at the exit from the waveguide with the TRIW increases by a factor of four [curves (*1*, *2*)] and six [curve (*3*)] times. After passing such a pulse through a dispersing element (in our case, through a diffraction grating with $D_g = -10^{-24}$ s^2 [15, 16]), it is possible to ensure its strong compression, while its peak power will increase by more than an order of magnitude. It follows from Fig. 3 that a broadband WP with a spectral width $\Delta\omega > 10^{12}$ s^{-1} can be effectively amplified while retaining its shape in a medium with a much smaller gain line width ($\Delta\omega_{\text{lin}} = 10^{11}$ s^{-1}). In this case, the pulse in the corresponding amplifier retains an almost linear chirp, which ensures its further effective compression on a diffraction grating to subpicosecond durations and kilowatt peak powers. Figure 3 shows that the amplification efficiency is determined by the value of the cubic (Kerr) nonlinearity R. In this case, at the attained powers, the amplification efficiency reaches some optimal values at $R < 10^{-3}$ W^{-1} m^{-1}. The effect of nonlinearity can be reduced by increasing the effective area S_{eff} (defocusing) of the modulated and amplified wave packet [15]. In this case, effective amplification of a pulse with a spectral width of more than 10 nm turns out to be possible (in the case of using amplifiers with TRIWs) even with an amplifier line width of less than 0.1 nm. On the other hand, the use of wider-band amplifiers is capable of generating pulses with a significantly higher energy than that of narrow-band amplifiers.

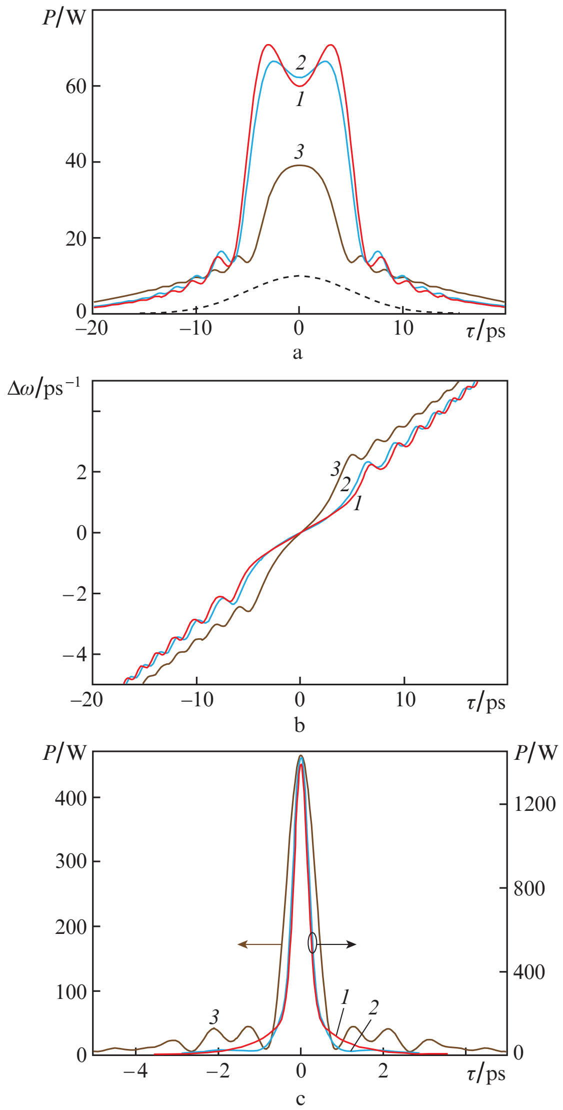

Figure 3. (Colour online) Dynamics of (a) the envelope, (b) spectrum of a pulse in a waveguide with a TRIW, as well as (c) its subsequent compression on a grating with a dispersion parameter $D_g = -10^{-24}$ s^2. Simulation parameters are $R = (1)\ 10^{-7}$, $(2)\ 10^{-5}$, and $(3)\ 10^{-3}$ W^{-1} m^{-1}; the width of the gain line is $\Delta\omega_{\text{lin}} = 10^{11}$ s^{-1}. Input pulse parameters (dashed curve) are: duration $\tau_0 = 10^{-11}$ s, and peak power $P_0 = 10$ W.

Figure 4 shows the results of calculating the dynamics of a broadband wave packet propagating in a modulated amplifier with the gain line width $\Delta\omega_{\text{lin}} = 10^{12}$ s^{-1}. Other parameters of the launched radiation and the medium are the same as in Fig. 3. It can be seen that the energy of the obtained pulses is much higher than in the previous case, and that the efficiency of the final compression of the pulses is determined by the value of the effective nonlinear parameter R: for the initial parameters used $g_0 \approx 10^{-2}$ m^{-1} and $P_0 = 10$ W, the degree of final compression for nonlinearities $R < 10^{-5}$ W^{-1} m^{-1} is virtually the same. With an increase in the pulse power (and an increase in nonlinear effects), the wave packet in such a medium is not preserved [curves (*3*)]. In contrast to the previous case, it follows from Fig. 4 that at the same values of nonlinearity R, the part of the pulse spectrum near the carrier frequency suffers most of all, and therefore the chirp cannot be considered linear in this region. Thus, we can assume that the technique of amplifying FM pulses in media with a much narrower gain line bandwidth is a good tool for their high-quality compression and amplification.

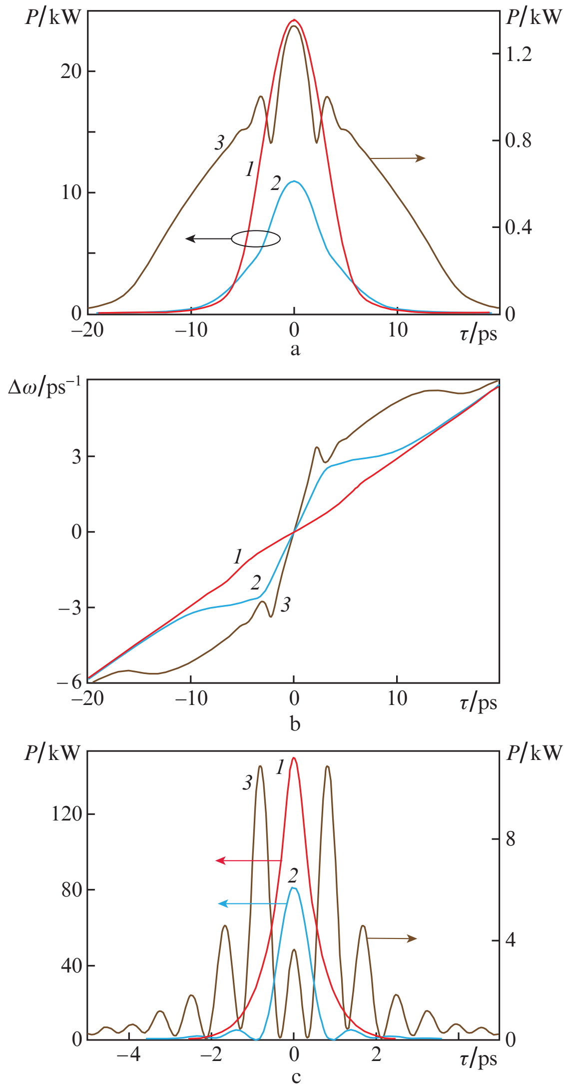

Figure 4. (Colour online) Dynamics of (a) the envelope, (b) spectrum of a pulse in a waveguide with a TRIW, as well as (c) its subsequent compression. Simulation parameters are $R = (1)\ 10^{-7}$, $(2)\ 10^{-5}$, and $(3)\ 10^{-3}$ W^{-1} m^{-1}; the width of the gain line is $\Delta\omega_{\text{lin}} = 10^{12}$ s^{-1}; other parameters are the same as in Fig. 1.

Summarising, we note that the possibility of defocusing the amplified and modulated beam can provide a large compression of laser radiation (since the nonlinearity coefficient is inversely proportional to the effective area of the beam [15, 16]), while the width of the gain line of the medium can be much less than the width of the pulse spectrum. The implemented TRIW stabilises the amplified FM pulse, providing an increase in its energy while maintaining its shape.

4. Conclusions

We have propose a scheme for synchronising a WGM-type wave packet tunnelling over the waveguide surface and a longitudinally travelling refractive-index wave. It has been shown that in active cylindrical waveguide modulators with a TRIW it is possible to provide spectral broadening and amplification of broadband FM pulses with a conserved shape and line width $\Delta\omega > 10^{13}$ s^{-1} inclusively when using relatively narrowband amplifiers with a gain line width of no more than $10^{11}-10^{12}$ s^{-1}. Using numerical simulations we have shown that in modulators with a TRIW, the peak power of amplified pulses can be increased by several orders of magnitude.

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Appendix

Let the dynamics of the considered WP be described by equation (4). If the mismatch between the velocities of the wave packet and the TRIW is taken to be small (i.e., $\delta\tau \leqslant 10^{-11}$ s for $|\Omega|\tau_{\rm p} \ll 1$), then we can assume that $\cos[\Omega(\tau - \delta\tau)] \approx 1 - \Omega^2(\tau - \delta\tau)^2/2$.

Passing in equation (4) to the notation $A(\xi) = \bar{A}(\xi) \times \exp(g_0\xi)$ and $R_1(\xi) = R\exp(2g_0\xi)$, and also assuming that $d_3 \approx 0$, we obtain

$$\frac{\partial\bar{A}}{\partial\xi} - \mathrm{i}\frac{d_2}{2}\frac{\partial^2\bar{A}}{\partial\tau^2} + \mathrm{i}R_1(z)\left(|\bar{A}|^2 - \tau_{\rm R}\frac{\partial|\bar{A}|^2}{\partial\tau}\right)\bar{A}$$
$$= \mathrm{i}m\beta\left(1 - \frac{1}{2}\Omega^2(\tau - \delta\tau)^2\right)\bar{A}, \qquad \text{(A1)}$$

or

$$\frac{\partial\bar{A}}{\partial\xi} - \mathrm{i}\frac{d_2}{2}\frac{\partial^2\bar{A}}{\partial\tau^2} + \mathrm{i}R_1(\xi)\left(|\bar{A}|^2 - \tau_{\rm R}\frac{\partial|\bar{A}|^2}{\partial\tau}\right)\bar{A}$$
$$= \mathrm{i}(S_0 + S_1\tau + S_2\tau^2)\bar{A}, \qquad \text{(A2)}$$

where $S_0 = m\beta(1 - \Omega^2\delta\tau^2/2)$; $S_1 = m\beta\Omega^2\delta\tau$; and $S_2 = -m\beta\Omega^2/2$.

In the case of a dynamic waveguide under consideration, the maximum of the refractive index of the TRIW corresponds to the modulation depth $m < 0$. We will seek the solution to (A2) in the form

$$\bar{A} = a(\xi,\tau)\exp(\mathrm{i}\Phi(\xi)) = a(\xi,\tau)\exp[\mathrm{i}(\phi_0(\xi)$$
$$+ \phi_1(\xi)\tau + \alpha(\xi)\tau^2)], \qquad \text{(A3)}$$

where $\Phi(\xi) = \phi_0(\xi) + \phi_1(\xi)\tau + \alpha(\xi)\tau^2$ and $a(\xi,\tau)$ are the WP phase and amplitude, respectively. Substituting (A3) into (A2), we obtain the expansion

$$\mathrm{e}^{\mathrm{i}\Phi}\left\{\frac{\partial a}{\partial\xi} + \mathrm{i}a\left(\frac{\partial\phi_0}{\partial\xi} + \tau\frac{\partial\phi_1}{\partial\xi} + \tau^2\frac{\partial\alpha}{\partial\xi}\right) - \mathrm{i}\frac{d_2}{2}\right.$$
$$\left.\times\left[\frac{\partial^2 a}{\partial\tau^2} + 2\mathrm{i}(\phi_1 + 2\alpha\tau)\frac{\partial a}{\partial\tau} + (2\mathrm{i}\alpha - \phi_1^2 - 4\tau^2\alpha^2 - 4\phi_1\alpha\tau)a\right]\right\}$$
$$+ \mathrm{i}R_1\mathrm{e}^{\mathrm{i}\Phi}\left(|a|^2 - \tau_{\rm R}\frac{\partial|a|^2}{\partial\tau}\right)a = \mathrm{i}\mathrm{e}^{\mathrm{i}\Phi}(S_0 + S_1\tau + S_2\tau^2)a \qquad \text{(A4)}$$

and grouping the terms, we obtain

$$\frac{\partial a}{\partial\xi} - \mathrm{i}\frac{d_2}{2}\frac{\partial^2 a}{\partial\tau^2} + \mathrm{i}R_1\left(|a|^2 - \tau_{\rm R}\frac{\partial|a|^2}{\partial\tau}\right)a$$
$$+ \mathrm{i}\tau^2\left(\frac{\partial\phi_2}{\partial\xi} + 2d_2\alpha^2\right)a + \mathrm{i}\tau\left(\frac{\partial\phi_1}{\partial\xi} + 2d_2\phi_1\alpha\right)a$$
$$- \mathrm{i}\frac{d_2}{2}\left[2\mathrm{i}\phi_1\frac{\partial a}{\partial\tau} + (2\mathrm{i}\alpha - \phi_1^2)a\right]$$
$$+ \mathrm{i}a\frac{\partial\phi_0}{\partial\xi} + 2d_2\alpha\tau\frac{\partial a}{\partial\tau} = \mathrm{i}(S_0 + S_1\tau + S_2\tau^2)a.$$

Equation (A4) should make sense for any time values; therefore, the terms containing constants and time factors τ, and τ^2 should add up to zero. Thus, we obtain equations for the phase parameters and the amplitude of the corresponding wave packet. First equation has the form

$$\frac{\partial\alpha}{\partial\xi} + 2d_2\alpha^2 = S_2 = -\frac{1}{2}m\beta\,\Omega^2. \qquad \text{(A5)}$$

For $d_2 > 0$ and $m < 0$, the waveguide mode is realised, i.e. the wave is pulled into the region of the maximum refractive index of the TRIW, and the solution for the WP chirp has the form:

$$\alpha(\xi) = \frac{\exp(2\sqrt{2S_2d_2}\,\xi) - 1}{\exp(2\sqrt{2S_2d_2}\,\xi) + 1}\sqrt{\frac{S_2}{2d_2}}. \qquad \text{(A6)}$$

With an effective length much less than the dispersion length, when $\xi \ll \tau_0^2/|d_2|$, relation (A6) is reduced to relation (6).

For the phase parameter ϕ_1, which determines the linear frequency shift, we obtain the second equation:

$$\frac{\partial\phi_1}{\partial\xi} + 2\alpha d_2\phi_1 = S_1. \qquad \text{(A7)}$$

The third equation is obtained for the amplitude parameter a:

$$\frac{\partial a}{\partial\xi} + (d_2\phi_1 + 2d_2\alpha\tau)\frac{\partial a}{\partial\tau} - \mathrm{i}\frac{d_2}{2}\frac{\partial^2 a}{\partial\tau^2}$$
$$+ \mathrm{i}R_1\left(|a|^2 - \tau_{\rm R}\frac{\partial|a|^2}{\partial\tau}\right)a = F(\xi)a, \qquad \text{(A8)}$$

where

$$F(\xi) = \mathrm{i}\left[S_0 - \frac{\partial\phi_0}{\partial\xi} + \frac{d_2}{2}(2\mathrm{i}\alpha - \phi_1^2)\right]$$
$$= \mathrm{i}\left(S_0 - \frac{\partial\phi_0}{\partial\xi} - \frac{d_2\phi_1^2}{2}\right) - d_2\alpha,$$

or, passing to the amplitude

$$\bar{a} = a\exp\left(\int_0^{\xi} F(\xi)\mathrm{d}\xi\right), \qquad \text{(A9)}$$

we obtain

$$\frac{\partial\bar{a}}{\partial\xi} + (\eta_1 + \eta_2\tau)\frac{\partial\bar{a}}{\partial\tau} - \mathrm{i}\frac{d_2}{2}\frac{\partial^2\bar{a}}{\partial\tau^2}$$
$$+ \mathrm{i}R_{\mathrm{eff}}\left(|\bar{a}|^2 - \tau_{\mathrm{R}}\frac{\partial|\bar{a}|^2}{\partial\tau}\right)\bar{a} = 0, \qquad \text{(A10)}$$

where $\eta_1(\xi) = d_2\phi_1$, and $\eta_2(\xi) = 2d_2\alpha$. In this case, the effective nonlinearity is expressed as

$$R_{\mathrm{eff}}(\xi) = R_1(\xi)\exp\left(2\int_0^{\xi}\mathrm{Re}[F(\xi)]\mathrm{d}\xi\right) = R_1(\xi)$$
$$\times\exp\left(-\int_0^{\xi}\eta_2(\xi)\mathrm{d}\xi\right) = R(\xi)\exp\left(2g_0\xi - \int_0^{\xi}\eta_2(\xi)\mathrm{d}\xi\right). \qquad \text{(A11)}$$

Since in our case we can assume with good accuracy that $\delta\tau \to 0$ and, as a consequence, $\phi_1 \to 0$, for $\bar{a}$ we have

$$\frac{\partial\bar{a}}{\partial\xi} + 2\alpha d_2\frac{\partial\bar{a}}{\partial\tau} - \mathrm{i}\frac{d_2}{2}\frac{\partial^2\bar{a}}{\partial\tau^2} + \mathrm{i}R_{\mathrm{eff}}\left(|\bar{a}|^2 - \tau_{\mathrm{R}}\frac{\partial|\bar{a}|^2}{\partial\tau}\right)\bar{a} = 0. \qquad \text{(A12)}$$

Passing in equation (A12) to new coordinates (length and normalised running time),

$$\xi = \xi,\ \tau' = \exp\left(-\int_0^{\xi}\eta_2(\xi)\mathrm{d}\xi\right)\tau, \qquad \text{(A13)}$$

we obtain an equation of the canonical form for $\bar{a}$ (the nonlinear Schrödinger equation with the Raman self-scattering parameter) [15]:

$$\frac{\partial\bar{a}}{\partial\xi} - \mathrm{i}\frac{d_2^{\mathrm{eff}}(\xi)}{2}\frac{\partial^2\bar{a}}{\partial\tau'^2} + \mathrm{i}R_{\mathrm{eff}}$$
$$\times\left[|\bar{a}|^2 - \exp\left(-\int_0^{\xi}\eta_2(\xi)\mathrm{d}\xi\right)\tau_{\mathrm{R}}\frac{\partial|\bar{a}|^2}{\partial\tau'}\right]\bar{a} = 0, \qquad \text{(A14)}$$

where

$$d_2^{\mathrm{eff}}(\xi) = \exp\left(-2\int_0^{\xi}\eta_2(\xi)\mathrm{d}\xi\right)d_2.$$

The performed analysis makes it possible to evaluate some important features of the dynamics of FM pulses in the conditions of interaction with the TRIW. Thus, from relations (A5), (A6) and (A14) it follows that in a normal-dispersion medium, in the approximation when we can assume that

$$R_{\mathrm{eff}}(\xi)P_0/\tau_{\mathrm{p}}'^2(\xi) < |m|\beta\Omega^2, \qquad \text{(A15)}$$

where

$$\tau_{\mathrm{p}}'(\xi) = \tau_{\mathrm{p}}(\xi)\exp\left(\int_0^{\xi}\eta_2(\xi)\mathrm{d}\xi\right)$$

is the normalised pulse duration, the pulse retains its shape, and its energy, duration, chirp and, as a consequence, the spectrum width increase [see (8)]. In this case, the frequency modulation rate (with a high degree of accuracy) remains linear.

It can also be seen that, since in our case $\eta_2(\xi) = 2d_2\alpha > 0$, the presence of the TRIW decreases the effect of the effective value of nonlinearity, which leads to additional stabilisation of the system with an increase in the energy of the modulated wave packet; in addition, the normalised pulse duration in a medium with Kerr nonlinearity and normal dispersion is further increased.

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