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Noise-oriented quantum optical gyrometry

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Abstract. **By the example of an optical gyroscope scheme, a new method for improving the accuracy of phase measurements is considered. In the rotation-recording Mach–Zehnder interferometer, a two-mode squeezed vacuum is used as an input state. This does not allow realising the traditional scheme, since the average value of the difference signal at the output is always zero. However, it is shown that information about the magnitude of the rotation angular velocity of the instrument reference frame is contained in the noise level of the difference signal. The possibility of reaching the Heisenberg limit of the measurement accuracy is demonstrated.**

Keywords: *optical gyroscope, two-mode squeezed vacuum, Mach–Zehnder interferometer, measurement accuracy.*

1. Introduction

It is known that the accuracy of determining a physical parameter (e.g., the phase φ) can be increased by repeating the measurement procedure and averaging the results. In this case, the variance of the measured parameter decreases as $1/\sqrt{N}$, where N is the number of repeated measurements. If, when measuring φ by quantum optical interferometry, a state with an average number of photons $\langle N\rangle$ is used and $\Delta\varphi = 1/\sqrt{\langle N\rangle}$, then this corresponds to the so-called standard quantum limit (SQL) of sensitivity. The simplest version, demonstrating SQL, is a scheme using a Mach–Zehnder interferometer (MZI) with a coherent Glauber state at the input. The complete measurement effectively decomposes into separate elementary procedures, each involving a single photon. With a more subtle use of quantum effects, e.g., when preparing states of a special type that have no classical analogues, it is possible to achieve the so-called Heisenberg limit (HL) of accuracy: $\Delta\varphi = 1/\langle N\rangle$, which is the goal of quantum metrology [1]. It is possible to overcome SQL in the MZI scheme with a squeezed vacuum state in the second input arm of the interferometer [2]. However, the Heisenberg limit is not reached in this case, since $\Delta\varphi = 1/\langle N\rangle^{3/4}$. One of the ways to achieve HL is to prepare entangled states of the input modes.

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The physical foundations of this approach are described in Refs [3, 4].

Among the highly non-classical radiation states, whose preparation technology is sufficiently developed, the state of a two-mode squeezed vacuum

$$|\mathrm{TMSV}\rangle = \sum_{n=0}^{\infty}\sqrt{p_n(r)}\,|n,n\rangle \qquad (1)$$

is known as a certain superposition of the Fock states of two modes. Here r is the squeezing parameter, and $p_n(r) = \cosh^{-2}(r)\tanh^{2n}(r)$. The properties of this state and the methods of its preparation are considered, e.g., in Ref. [5]. Unfortunately, the direct use of $|\mathrm{TMSV}\rangle$ in the traditional scheme of phase measurements from the average intensity difference of the output modes is impossible, since the average difference is insensitive to phase shifts and is simply zero [6]. This circumstance prompts the search for non-standard measuring procedures at the MZI output. Anisimov et al. [7] proposed to measure the parity of the number of photons in one of the interferometer output modes. It was shown that this allows the Heisenberg limit to be reached and even surpassed in a certain sense:

$$\Delta\varphi = 1/\sqrt{\langle N\rangle(\langle N\rangle + 2)} < 1/\langle N\rangle.$$

Here we propose an alternative way of using state (1) in phase measurements oriented at the noise level of the intensity difference at the MZI output. It is shown that under certain conditions, HL can be reached. The possibility of obtaining such a level of accuracy, as well as the required range of the parameter r entering $|\mathrm{TMSV}\rangle$, is determined by the magnitude of the measured phase. The results obtained are applicable to measuring phases of different nature. We have chosen a specific actual problem of gyrometry, namely, measuring the angular velocity of the Earth's rotation around its axis by determining the relevant Sagnac phase. The corresponding scheme is considered in Section 2, where the existence of a regime with the attainment of the Heisenberg limit is proved. In Section 3, the limits of the accuracy of phase measurements, which are attainable with a two-mode squeezed vacuum state, are studied from a general point of view. The dependence of the Cramér–Rao bound on the parameter r is obtained, and the optimality of the selected regime is shown. Conclusions summarise the results and outline the immediate goals of further research.

2. Gyroscope model

Traditional optical gyroscopes (gyrometers) measure the Sagnac phase generated by the rotation of the instrument's frame of reference [8]. We used the data available in the literature on the parameters of modern aircraft fibre-optic gyrometers [9]. Instead of the traditional scheme of a gyroscope with a Sagnac interferometer, we use a scheme based on a Mach–Zehnder interferometer (Fig. 1) with a geometric structure similar to that of a Sagnac interferometer [9]. We assume that the MZI is formed by two optical fibres with a length of $L = 10^5$ cm, wound oppositely on a coil with a radius of R = 5 cm, the radiation frequency being $\omega = 10^{15}$ s^{-1}. The angular velocity of the Earth's rotation is $\approx 7.27 \times 10^{-5}$ s^{-1}. The corresponding Sagnac phase is

$$\Phi_{\mathrm{S}} = \frac{4RL\Omega\omega}{c^2} \approx 16.17 \times 10^{-5}, \tag{2}$$

where c is the speed of light. The phases $\pm\phi$ in the internal arms of the interferometer (see Fig. 1) with the indicated direction of rotation are equal to $\pm\Phi_{\mathrm{S}}/4$.

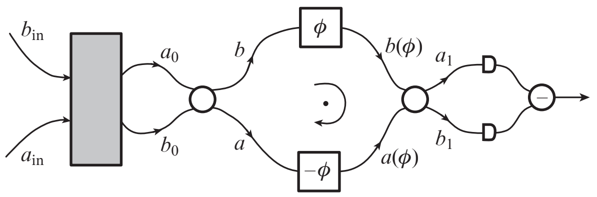

Figure 1. Schematic of a gyroscope based on a Mach–Zehnder interferometer. The rectangular box denotes a nonlinear medium where a two-mode squeezed vacuum state is generated.

In the Heisenberg picture, which is used to analyse the gyrometer operation, the operators of photon modes are transformed in different parts of the scheme shown in Fig. 1. The nonlinear medium performs the transformation

$$\hat{a}_0 = \cosh(r)\,\hat{a}_{\mathrm{in}} + \sinh(r)\,\hat{b}^{\dagger}_{\mathrm{in}}, \quad \hat{b}_0 = \cosh(r)\,\hat{b}_{\mathrm{in}} + \sinh(r)\,\hat{a}^{\dagger}_{\mathrm{in}}. \tag{3}$$

The state of radiation in the Heisenberg picture is vacuum with respect to the operators $\hat{a}_{\mathrm{in}}$ and $\hat{b}_{\mathrm{in}}$ and remains unchanged. With respect to the pair $\hat{a}_0$ and $\hat{b}_0$, this state turns out to be the state $|\mathrm{TMSV}\rangle$ from Eqn (1). The vectors $|n,n\rangle$ entering Eqn (1) are actually $|n\rangle_{a_0}\otimes|n\rangle_{b_0}$. The beam splitters in the interferometer are assumed to be balanced: $\hat{a} = (\hat{a}_0 + \hat{b}_0)/\sqrt{2}$, and $\hat{b} = (-\hat{a}_0 + \hat{b}_0)/\sqrt{2}$. The ordered pairs $(\hat{b}_1, \hat{a}_1)$ and $(\hat{a}(\phi), \hat{b}(\phi))$ are related in a similar way, where $\hat{a}(\phi) = \hat{a}\exp(\mathrm{i}\phi)$ and $\hat{b}(\phi) = \hat{b}\exp(-\mathrm{i}\phi)$. This gives the form of relation between operators at the MZI input and output:

$$\hat{a}_1 = \cos(\phi)\,\hat{a}_0 + \mathrm{i}\sin(\phi)\,\hat{b}_0, \quad \hat{b}_1 = \mathrm{i}\sin(\phi)\,\hat{a}_0 + \cos(\phi)\,\hat{b}_0. \tag{4}$$

Relations (3) and (4) allow the operator of the output difference signal $\hat{n}_1 = \hat{a}^{\dagger}_1\hat{a}_1 - \hat{b}^{\dagger}_1\hat{b}_1$ to be expressed in terms of $\hat{a}_{\mathrm{in}}$, $\hat{b}_{\mathrm{in}}$ and their Hermitian conjugates:

$$\hat{n}_1 = \cos(2\phi)\,\hat{n}_{\mathrm{in}} + \mathrm{i}\sin(2\phi)\,\hat{m}. \tag{5}$$

Here

$$\hat{n}_{\mathrm{in}} \doteq \hat{a}^{\dagger}_{\mathrm{in}}\hat{a}_{\mathrm{in}} - \hat{b}^{\dagger}_{\mathrm{in}}\hat{b}_{\mathrm{in}},$$

$$\hat{m} \doteq \cosh(2r)(\hat{a}^{\dagger}_{\mathrm{in}}\hat{b}_{\mathrm{in}} - \hat{b}^{\dagger}_{\mathrm{in}}\hat{a}_{\mathrm{in}}) + \tfrac{1}{2}\sinh(2r)(\hat{a}^{\dagger 2}_{\mathrm{in}} - \hat{b}^{\dagger 2}_{\mathrm{in}} - \hat{a}^2_{\mathrm{in}} + \hat{b}^2_{\mathrm{in}}).$$

All averagings $\langle\ldots\rangle$ are then calculated over the vacuum state of the modes of $\hat{a}_{\mathrm{in}}$ and $\hat{b}_{\mathrm{in}}$. The obvious result $\langle\hat{n}_1\rangle = 0$ corresponds to the impossibility of using $|\mathrm{TMSV}\rangle$ (see Section 1) within the framework of the traditional measurement scheme orientated at the dependence of $\langle\hat{n}_1\rangle$ on ϕ.

The quadratic variance of the difference signal (noise) appears for nonzero values of ϕ:

$$\langle(\Delta\hat{n}_1)^2\rangle = \langle\hat{n}_1^2\rangle = -\sin^2(2\phi)\,\langle\hat{m}^2\rangle = \sinh^2(2r)\sin^2(2\phi). \tag{6}$$

To estimate the measurement accuracy, it is necessary to know the variance of the noise level. In the expression for the mean value of $\hat{n}_1$ to the fourth power,

$$\langle\hat{n}_1^4\rangle = \sin^4(2\phi)\,\langle\hat{m}^4\rangle - \cos^2(2\phi)\sin^2(2\phi)\,\langle\hat{m}\hat{n}^2_{\mathrm{in}}\hat{m}\rangle - \mathrm{i}\cos(2\phi)\sin^3(2\phi)(\langle\hat{m}^2\hat{n}_{\mathrm{in}}\hat{m}\rangle + \langle\hat{m}\hat{n}_{\mathrm{in}}\hat{m}^2\rangle), \tag{7}$$

only the first two terms turn to be nonzero. As a result of rather lengthy calculations, we arrive at the relation

$$\langle(\Delta\hat{n}_1^2)^2\rangle = 4\cosh(4r)\sinh^2(2r)\sin^4(2\phi) + \sinh^2(2r)\sin^2(4\phi). \tag{8}$$

Two terms on the right-hand side of Eqn (8) demonstrate a different dependence on r and ϕ. In the case of small values of ϕ, which is of primary interest, the first term is proportional to ϕ^4, and the second to ϕ^2. The first term contains the factor $\cosh(4r)$, which can compensate for the additional second power of ϕ. If, however, the condition

$$\cosh(4r) \ll \frac{1}{4\phi^2} \tag{9}$$

is satisfied, the second term in Eqn (8) turns out to be the major one. In this case, from the relation $(\mathrm{d}\langle\hat{n}_1^2\rangle/\mathrm{d}\phi)\Delta\phi = \sqrt{\langle(\Delta\hat{n}_1^2)^2\rangle}$ it follows that

$$\Delta\phi = \frac{1}{2\sinh(2r)}. \tag{10}$$

The average number of photons $\langle N\rangle$ in the state $|\mathrm{TMSV}\rangle$ is $2\sinh^2(r)$. For $r \geqslant 1$

$$\Delta\phi \approx \frac{1}{2\langle N\rangle}. \tag{11}$$

If we take into account that the magnitude of the equivalent phase shift when it is localised in one arm of the MZI is equal to 2ϕ, then relation (11) is indicates that the Heisenberg limit of accuracy has been reached.

3. Accuracy limit of phase measurements with |TMSV⟩

Estimate (10) for the accuracy of ϕ measurement should be compared with the corresponding Cramér–Rao bound [10]. This provides information about the potential capabilities of the $|\mathrm{TMSV}\rangle$ state for measuring the phase in quantum metrology, not only in the proposed scheme, but also in any other one. As far as we know, there is no such analysis in the literature. It is convenient to start by calculating the classical Fisher information for the Fock state $|n,n\rangle$ at the MZI input. Using the relations inverse to (4), we can find the form of the state at the output of the interferometer:

$$|\Psi_\phi(n)\rangle = \frac{1}{n!}\left(\cos(\phi)\hat{a}_1^\dagger + \mathrm{i}\sin(\phi)\hat{b}_1^\dagger\right)^n \times\left(\cos(\phi)\hat{b}_1^\dagger + \mathrm{i}\sin(\phi)\hat{a}_1^\dagger\right)^n|0,0\rangle. \qquad (12)$$

We are interested in the probability of fixing a certain absolute value of the difference $2|n-k|$ of the number of photons in modes a_1 and b_1. This probability is the sum (for $n \neq k$) of two equal probabilities $P_\phi^{(n)}(k)$ and $P_\phi^{(n)}(2n-k)$ of detecting the modes in the states $|k, 2n-k\rangle$ and $|2n-k, k\rangle$, respectively. From Eqn (12) we obtain an expression for $P_\phi^{(n)}(k)$:

$$P_\phi^{(n)}(k) = \frac{k!(2n-k)!}{(n!)^2} \times\left(\sum_{k'=0}^{k} C_n^{k'} C_n^{k-k'}(-1)^{k'}\cos^{n-k+2k'}(\phi)\sin^{n+k-2k'}(\phi)\right)^2. \qquad (13)$$

The classical Fisher information for the state $|\Psi_\phi(n)\rangle$, as the quadratic variance of the corresponding risk function [10], is defined as

$$F_{\mathrm{cl}}^{(n)} = 4\sum_{k=0}^{2n}\left[\frac{k!(2n-k)!}{(n!)^2}\left(\sum_{k'=0}^{k} C_n^{k'} C_n^{k-k'}(-1)^{k'}\cos^{n-k+2k'-1}(\phi) \times\sin^{n+k-2k'-1}(\phi)\left(n\cos(2\phi)+k-2k'\right)\right)^2\right]. \qquad (14)$$

Fisher's quantum information is not related to a certain type of measurement. For a pure state, such as $|\Psi_\phi(n)\rangle$, it takes the form [10]: $F_{\mathrm{q}}^{(n)} = 4\langle\Psi_\phi'(n)|\Psi_\phi'(n)\rangle + 4\langle\Psi_\phi'(n)|\Psi_\phi(n)\rangle^2$, where the prime denotes differentiation with respect to ϕ. In this expression, only the first term turns out to be nonzero. Its calculation leads to an expression identical to (14), i.e., $F^{(n)} \doteq F_{\mathrm{q}}^{(n)} = F_{\mathrm{cl}}^{(n)}$.

Since when the difference signal is detected at the MZI output, information about the total number of photons inevitably appears, the various terms of the superposition (1) make an independent contribution to the complete Fisher information:

$$F = \sum_{n=0}^{\infty} p_n F^{(n)}, \qquad (15)$$

and F turns out to be a function only of r and does not depend on ϕ. This means that the Cramér–Rao inequality $\Delta\phi \geqslant F^{-1/2}$ gives a uniform in ϕ lower bound on the measurement inaccuracy. However, with respect to r, there is an optimum $r_{\mathrm{opt}} \approx 3.8$ (Fig. 2), near which inequality (9) is fulfilled. At the same time, obviously, the Heisenberg limit regime takes place.

4. Conclusions

1. A measurement scheme is proposed, in which a two-mode squeezed vacuum state is used. This state arises in a spontaneous parametric process. The technology for obtaining the state $|\mathrm{TMSV}\rangle$ is already quite well developed. The single-mode squeezed vacuum state, also used in quantum metrology [2], is, in a sense, a secondary product and is obtained by transforming the two-mode squeezed vacuum state [5].

2. The coherent nature of the state $|\mathrm{TMSV}\rangle$, which makes it an entangled state of two input modes, turns out to be unimportant. The separable state

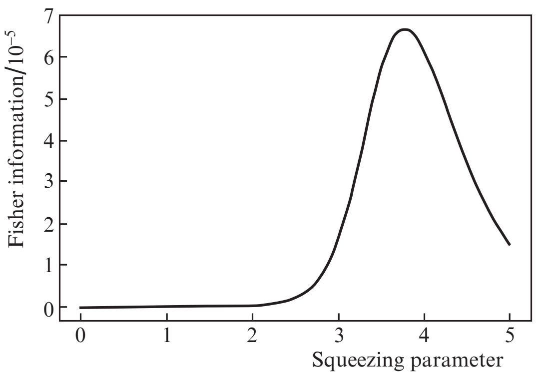

Figure 2. Dependence of Fisher information on the squeezing parameter r.

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle_{a_0}\langle n| \otimes |n\rangle_{b_0}\langle n|$$

could be used as well. For this reason, the proposed scheme is resistant to possible disruptions in the relative phases of the terms in Eqn (1) for $|\mathrm{TMSV}\rangle$. The main necessary property of the input state is the equality of the number of photons in both modes.

3. The orientation towards measuring the noise level in the output difference signal seems preferable to measuring the parity of the number of photons in one of the outputs, which is proposed in Ref. [7]. Note also that in our scheme there is an effect of a slight excess above the Heisenberg limit of accuracy, similar to that stated in [7], since in addition to the main term demonstrating the Heisenberg limit, the denominator of Eqn (10) initially contained positive terms, which were 'discarded' due to their relative smallness.

The immediate goal of further research is to calculate the correlation function $\langle\Omega(t_2)\Omega(t_1)\rangle$ of the measured angular velocity values. This will allow determining the Allan deviation calculated from the measurements of the relative bearing.

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