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Itô–Stratonovich dilemma in the problem of laser cooling of atoms: limits of applicability of the semiclassical approximation

A.A. Kirpichnikova, O.N. Prudnikov, A.V. Taichenachev, V.I. Yudin

Abstract. **We report a comparative analysis of the limits of laser cooling of atoms in light fields both on the basis of a numerical solution of the quantum kinetic equation for the atomic density matrix, which makes it possible to accurately take into account quantum recoil effects during the interaction of atoms with field photons, and within the framework of the semiclassical approach using various forms of the Fokker–Plank equation. The analysis allows one to outline the limits of applicability of the semiclassical approach, as well as to choose an unambiguous form for introducing the diffusion contribution to the Fokker–Planck equation in the semiclassical description of atomic kinetics.**

Keywords: *laser cooling of atoms, semiclassical approach, quantum recoil effects.*

1. Introduction

Cold atoms play a key role in modern atomic physics and are the main tool for high-precision measurements. The most impressive progress has been made in the field of frequency standards based on cold atoms and ions with a relative uncertainty of 10^{-18} and better [1–6]. High-precision atomic interferometers are widely used in modern precision measurements for fundamental and applied research, such as measurements of the fine structure constant [7, 8], Newton's gravitational constant [9, 10], tests of general relativity [11], and gravity measurements [12, 13].

Laser cooling of atoms by a resonant light field is based on the fundamental processes of exchange of momentum and energy of atoms with field photons during their interaction. To date, there are many approaches to describe the problem of laser cooling, which methodologically can be divided into quantum and semiclassical. Quantum approaches (see, for example, [14–20]) make it possible to accurately take into account the processes related to changes in the internal and translational degrees of freedom of atoms and their mutual correlation in the interaction with field photons; however, they require significant computational resources.

Semiclassical approaches are based on a number of approximations and allow one to describe the process of laser cooling of atoms in terms of the force acting on atoms from the side of the light field and its fluctuations caused by a jump change in the momentum of atoms upon interaction with field photons (see, for example, [21–23]). In some cases they make it possible to obtain analytical expressions for the forces, diffusion coefficients, and limiting temperatures of laser cooling of atoms [21–25]. Force fluctuations lead to heating and actually determine the minimum achievable atomic temperatures. In analysing the kinetics of atoms in light fields, these fluctuations are described by the diffusion coefficient D in the Fokker–Planck (FP) equation for the distribution function of atoms in the phase space W [26]. Statistical approaches to solving the FP equation, based on the averaging of different trajectories of atoms, model these fluctuations by adding a Langevin equation to the force of a random momentum-dependent variable [26–29].

Because in the processes of absorption/emission of field photons the atomic momentum p instantly changes to a small but finite value $|p| = \hbar k$ (the momentum of the field photon), and the force acting on the atom depends on the speed of the atom, the question arises: At which value of the momentum should a random addition to the force be determined? This uncertainty in the choice of the momentum value is called the Itô–Stratonovich dilemma [26, 27, 30, 31], and in the framework of the FP equation it determines the differences in the form of the diffusion contribution [26]:

$$\frac{\partial^2}{\partial p^2}(DW) \quad \text{or} \quad \frac{\partial}{\partial p}\left(D\frac{\partial}{\partial p}W\right).$$

Note that under the conditions of the semiclassical description of the kinetics of atoms in light fields, namely, at an extremely small value of the photon recoil momentum to the width of the momentum distribution of atoms, $\hbar k \ll \Delta p$, as well as the smallness of the recoil energy received by the atom in the processes of absorption/emission of field photons $\hbar\omega_{\mathrm{R}} = \hbar^2 k^2/2M$ (M is the mass of the atom) to the natural linewidth of the resonant optical transition γ used for laser cooling (i.e., under conditions of a small recoil parameter $\varepsilon_{\mathrm{R}} = \omega_{\mathrm{R}}/\gamma \ll 1$), a random addition to the force, determined by the diffusion coefficient, has a weak dependence on velocity, and therefore, the variability in the choice of the diffusion

A.A. Kirpichnikova Institute of Laser Physics, Siberian Branch, Russian Academy of Sciences, prosp. Akad. Lavrent'eva 15B, 630090 Novosibirsk, Russia; e-mail: kirpichnikovaaa@gmail.com;
O.N. Prudnikov, A.V. Taichenachev Institute of Laser Physics, Siberian Branch, Russian Academy of Sciences, prosp. Akad. Lavrent'eva 15B, 630090 Novosibirsk, Russia; Novosibirsk State University, ul. Pirogova 1, 630090 Novosibirsk, Russia; e-mail: oleg.nsu@gmail.com;
V.I. Yudin Institute of Laser Physics, Siberian Branch, Russian Academy of Sciences, prosp. Akad. Lavrent'eva 15B, 630090 Novosibirsk, Russia; Novosibirsk State University, ul. Pirogova 1, 630090 Novosibirsk, Russia; Novosibirsk State Technical University, prosp. Karla Marksa 20, 630073 Novosibirsk, Russia; e-mail: viyudin@mail.ru

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term in the FP equation has an insignificant effect on the results of numerical analysis [31, 32]. However, the real values of the recoil energy for atoms cooled by a light field in combination with the use of narrow optical transitions do not always correspond to the condition of the extreme smallness of the parameter ε_R [33]. Also, under conditions of sub-Doppler laser cooling, the temperature can reach several recoil energies, which leads to rather narrow distributions of atoms in the momentum space and is characterised by sharp dependences of the force and diffusion coefficient on the atomic velocity [34, 35]. Under these conditions, the differences in the choice of the diffusion contribution in the FP equation become most significant.

In this paper, we investigate the Itô–Stratonovich dilemma by directly comparing the results for the problem of laser cooling of atoms, obtained in the framework of various semiclassical approaches to the Itô–Stratonovich problem, with the results obtained on the basis of the exact quantum approach [18–20]. The performed analysis of the kinetics of atoms with different recoil parameters ε_R made it possible to outline the limits of applicability of the semiclassical approach to the problem of laser cooling of atoms.

2. Itô vs. Stratonovich dilemma in the problem of laser cooling

The kinetics of atoms in an external light field is described by the quantum kinetic equation (QKE) for the atomic density matrix

$$\frac{\partial}{\partial t}\hat{\rho} = -\frac{\mathrm{i}}{\hbar}[\hat{H},\hat{\rho}] + \hat{\Gamma}\{\hat{\rho}\}, \tag{1}$$

containing contributions determined by the Hamiltonian $\hat{H}$ of an atom in an external electromagnetic field and the non-Hamiltonian evolution $\hat{\Gamma}\{\hat{\rho}\}$ as a result of decoherence of quantum states of an atom upon interaction with the external environment or as a result of spontaneous emission of photons (see, e.g., [33]). The presence of two main conditions – the smallness of the recoil momentum to the momentum distribution of atoms, $\hbar k/\Delta p \ll 1$, and the smallness of the parameter $\varepsilon_R = \omega_R/\gamma \ll 1$ (it is determined, among other things, by the mass of the atom and the natural linewidth of the optical transition used for laser cooling) – makes it possible to separate fast processes of ordering in terms of internal degrees of freedom from slow processes associated with the translational motion of an atom and reduce Eqn (1) to the FP equation for the distribution function of atoms in the phase space $W(r,v,t) = \mathrm{Tr}\{\hat{\rho}(r,v,t)\}$ (see, for example, [22, 36–39]). Here the trace is taken over the internal degrees of freedom of the atom. For a one-dimensional laser cooling problem, when the light field used for laser cooling is a combination of counterpropagating monochromatic waves with a frequency ω along the z axis with polarisations $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$,

$$\boldsymbol{E}(z,t) = \boldsymbol{e}_1 E_0 \mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{-\mathrm{i}\omega t} + \boldsymbol{e}_2 E_0 \mathrm{e}^{-\mathrm{i}kz}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{c.c.}, \tag{2}$$

the FP equation describing the kinetics of atoms in the semiclassical approximation is written in the form

$$\left(\frac{\partial}{\partial t} + \frac{p}{M}\frac{\partial}{\partial z}\right)W(z,p,t) = -\frac{\partial}{\partial p}F(z,p)\,W(z,p,t) + \frac{\partial^2}{\partial p^2}D(z,p)\,W(z,p,t) \tag{3}$$

(for example, in [21–23, 36, 38–41]) or in the form

$$\left(\frac{\partial}{\partial t} + \frac{p}{M}\frac{\partial}{\partial z}\right)W(z,p,t) = -\frac{\partial}{\partial p}F(z,p)\,W(z,p,t) + \frac{\partial}{\partial p}D(z,p)\frac{\partial}{\partial p}W(z,p,t) \tag{4}$$

(see, for example, [32, 42, 43]). The kinetic coefficients of the FP equation are $F(z,p)$, i.e. the force acting on the atom, and $D(z,p)$, i.e. the diffusion coefficient divided into the sum of contributions. These are the contributions of induced diffusion $D^{(\mathrm{i})}(z,p)$ associated with the fluctuation of the force in the processes of stimulated absorption/emission of field photons, and spontaneous diffusion $D^{(\mathrm{s})}(z,p)$ associated with the fluctuation of the momentum of atoms during spontaneous emission of photons [22, 36–39].

The differences between these forms of writing the FP equation are most clearly manifested in statistical approaches that simulate laser cooling of atoms in light fields, where the distribution function is obtained by statistical averaging over various trajectories of atoms, which are determined by the classical equations of motion

$$\frac{\mathrm{d}p}{\mathrm{d}t} = F(p,z) + \sqrt{2D(z,p)}\,\xi(t),\ \frac{\mathrm{d}z}{\mathrm{d}t} = p/M \tag{5}$$

with noise addition to force $\delta F = \sqrt{2D(z,p)}\,\xi(t)$. The latter is determined by a random variable $\xi(t)$, which has a normal distribution with zero mean $\langle\xi(t)\rangle$ and the correlation function

$$\langle\xi(t)\xi(t')\rangle = \delta(t-t'). \tag{6}$$

From a mathematical point of view, the equation of motion (5) with a δ-correlated noise term is not completely defined [26, 27]. Since the diffusion coefficient D, which determines the fluctuation of the force, depends on the velocity of the atom, the average value of the stochastic contribution $\langle\sqrt{2D(p)}\,\xi(t)\rangle$ is not equal to zero, which leads to a noise shift of the particle trajectory. In this case, it is necessary to determine at what value of the momentum the noise addition to the force δF is determined in the equation of motion (5): before adding a random variable (a time jump change in momentum), which corresponds to equation (3), or after adding a random contribution, which corresponds to (4). This uncertainty is known as the Itô–Stratonovich dilemma [26, 27, 30, 31].

Note that the differences in the forms of writing (3) and (4) can be reduced to the so-called noise or quantum correction to the force

$$\delta f(z,p) = \frac{\partial}{\partial p}D(z,p), \tag{7}$$

which, however, under the conditions of applicability of the semiclassical approach, is extremely small, $\delta f/F \simeq \varepsilon_R \ll 1$. Indeed, the FP equation can be obtained directly from the QKE for the atomic density matrix (1) by expanding the field photon recoil momentum in smallness, i.e., i.e. in the param-

eter $\hbar k/\Delta p$ up to second order terms. Under the conditions of the extreme smallness of the parameter ε_R ($\varepsilon_R \ll 1$), both approaches are equivalent, since the contributions of $\sim\varepsilon_R$ are neglected in the process of QKE reduction to the FP equation [22, 36–39]. However, if the recoil parameter ε_R is not sufficiently small, the differences between the two approaches can be significant. In this case, as noted in [42, 43], the results of Eqn (4) in some cases, neglecting the effects of localisation of atoms in the optical potential, are in better agreement with the results of the quantum approach, and for describing laser cooling, expression (4) looks more preferable . In Section 3, we performed a detailed comparison of the results of the semiclassical representation of the kinetics of atoms, obtained taking into account the nonlinear dependence of the kinetic coefficients of the FP equation, force, and diffusion on the velocity and also taking into account the localisation of atoms in the optical potential, with the results obtained on the basis of the numerical solution of the QKE for atomic density matrix (1) for atoms with different parameters of ε_R. This comparative analysis will make it possible to single out the form of the FP equation that best corresponds to the results of quantum calculations, as well as to outline the limits of applicability of the semiclassical approach.

3. Comparative analysis of semiclassical and quantum approaches

Let us consider the problem of laser cooling of atoms in a monochromatic field of a light wave (2), resonant to a closed optical transition $J_g \to J_e$, where J_g and J_e are the total angular momenta of the levels of the ground (g) and excited (e) states. Polarisations of counterpropagating waves, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, determine the spatial configuration of the cooling field. For example, orthogonally polarised counterpropagating waves form the configurations of light fields most used in laser cooling problems: lin $\perp$ lin configuration of the light field formed by counterpropagating waves with orthogonal linear polarisations, for example with $\boldsymbol{e}_1 = \boldsymbol{e}_x$, $\boldsymbol{e}_2 = \boldsymbol{e}_y$, and $\sigma_+ - \sigma$ configuration formed by counterpropagating waves with orthogonal circular polarisations $\boldsymbol{e}_1 = \boldsymbol{e}_+$, $\boldsymbol{e}_2 = \boldsymbol{e}_-$. Cooling in fields generated by waves with the same polarisations $\boldsymbol{e}_1 = \boldsymbol{e}_2$ can in some cases [25, 39] be reduced to the results of a simple two-level model with levels that are nondegenerate with respect to the angular momentum projection.

The evolution of an ensemble of low density atoms, for which interatomic interactions can be neglected, is described by the QKE in the single atom approximation (1). The Hamiltonian $\hat{H}$ is split into the sum of the contributions:

$$\hat{H} = \frac{\hat{p}^2}{2M} + \hat{H}_0 + \hat{V}, \tag{8}$$

where the first term is the kinetic energy operator; $\hat{H}_0 = -\hbar\delta\hat{\Pi}_e$ is the Hamiltonian of a free atom in the rotating wave approximation; $\delta = \omega - \omega_0$ is the detuning of the field frequency ω from the atomic transition frequency ω_0; and

$$\hat{\Pi}_e = \sum_{\mu} |J_e,\mu\rangle\langle J_e,\mu| \tag{9}$$

is the projection operator on the excited state levels $|J_e, \mu\rangle$, characterised by the total angular momentum J_e and the projection of the angular momentum μ onto the quantisation axis. The last term $\hat{V}$ in (8) describes the interaction of the atom with field (2). If field (2) is resonant to an electrodipole transition, the operator $\hat{V}$ takes the form

$$\hat{V} = \hat{V}_1 \exp(ikz) + \hat{V}_2 \exp(-ikz), \tag{10}$$

where

$$\hat{V}_n = \frac{\hbar\Omega}{2}(\hat{d}\boldsymbol{e}_n) = \frac{\hbar\Omega}{2}\sum_{\sigma=0,\pm1}\hat{d}_\sigma e_n^\sigma \quad (n = 1,2),$$

and is determined by the polarisation vectors of the counterpropagating waves and the vector operator of the dipole interaction $\hat{d}$, whose matrix components in the circular basis are expressed in terms of the Clebsch–Gordan coefficients; Ω is the Rabi frequency (see, for example, [33]).

Note that the stationary solution of QKE (1) for the atomic density matrix for a given polarisation configuration of the light field is characterised by three dimensionless parameters: the recoil parameter ε_R, the ratio of the field detuning to the natural linewidth δ/γ, and the ratio of the Rabi frequency to the natural linewidth Ω/γ. Instead of the last parameter, which determines the light field intensity, it is convenient to use a quantity that characterises the depth of the optical level shift U_0 and depends both on the light field intensity and detuning δ:

$$U_0 = \frac{\hbar|\delta|}{3}\frac{|\Omega|^2}{(\delta^2 + \gamma^2/4)}. \tag{11}$$

In particular, in the limit of low intensity of the cooling field, $|\Omega|^2/(\delta^2 + \gamma^2/4) \ll 1$, and in the secular approximation, $|\delta| \gg \gamma$, the light shift U_0 remains the only universal parameter characterising the stationary QKE solution [17]. We will also use this parameter to describe the laser field intensity, which, in particular, will allow us to compare the presented results with the previously obtained results [17] for the lin $\perp$ lin configuration of the light field in the framework of the secular approximation.

Within the framework of semiclassical approaches, the problem of atomic kinetics described by QKE (1) for the atomic density matrix $\hat{\rho}$, which contains complete information about the evolution of both internal and translational degrees of freedom, is reduced to the FP equation for the distribution function of atoms in the phase space W in one of selected forms [(3) or (4)]. The kinetic coefficients for the FP equation are obtained by expanding QKE (1) for the density matrix in terms of the small parameter $\hbar k/\Delta p$, i.e. the smallness of the ratio of the recoil momentum $\hbar k$ to the momentum distribution of atoms in the phase space Δp. The expansion of the QKE for the one-dimensional problem has the form (see, e.g., [39])

$$\left(\frac{\partial}{\partial t} + \frac{p}{M}\frac{\partial}{\partial z}\right)\hat{\rho}(z,p) = \hat{L}^{(0)}\{\hat{\rho}(z,p)\}$$

$$+ \hbar k\frac{\partial}{\partial p}\hat{L}^{(1)}\{\hat{\rho}(z,p)\} + \hbar^2k^2\frac{\partial^2}{\partial p^2}\hat{L}^{(2)}\{\hat{\rho}(z,p)\}\ldots \tag{12}$$

In particular, in the zeroth order in recoil effects, $\hbar k/\Delta p$ leads to the Bloch equation

$$\left(\frac{\partial}{\partial t}+\frac{p}{M}\frac{\partial}{\partial z}\right)\hat{\sigma}(z,p,t)=\hat{L}^{(0)}\{\hat{\sigma}(z,p,t)\} \tag{13}$$

for the density matrix $\hat{\sigma}$ under the normalisation condition $\mathrm{Tr}\{\sigma\}=1$, which describes the evolution of the internal degrees of freedom of the atom. The form of the functionals $\hat{L}^{(i)}$ $(i=0,1,2)$ of the QKE expansion is given in [39]. Accordingly, the expression for the force acting on an atom in a light field,

$$F(z,p)=\mathrm{Tr}\{\hat{L}^{(1)}\{\hat{\sigma}(z,p)\}\}=\mathrm{Tr}\{\hat{F}(z)\sigma(z,p)\}, \tag{14}$$

is the quantum mechanical average of the force operator $\hat{F}(z)=-\partial\hat{V}(z)/\partial z$. Here the trace is taken over the internal states of the atom, and $\sigma(z,p)$ is the stationary solution of Eqn (13).

The diffusion coefficient is represented as the sum of two terms, $D=D^{(\mathrm{s})}+D^{(\mathrm{i})}$. The spontaneous diffusion coefficient

$$D^{(\mathrm{s})}(z,p)=\mathrm{Tr}\{\hat{L}^{(2)}\{\hat{\sigma}(z,p)\}\} \tag{15}$$

is due to the recoil effect during spontaneous emission of field photons; the induced diffusion coefficient $D^{(\mathrm{i})}$ is determined by the fluctuation of the force acting on the atom in the processes of stimulated absorption/emission of field photons (see, for example, [22, 39]). In the zeroth order in velocity, the expression for $D^{(\mathrm{i})}$ can be obtained using the first correction matrix method proposed by us [39]. Taking into account the nonlinear dependence on velocity for slow atoms, the expression for $D^{(\mathrm{i})}$ can be obtained using the approach presented in [36, 37]:

$$D^{(\mathrm{i})}(z,p)=-\frac{1}{2}\mathrm{Tr}\{\hat{F}(z)\hat{\eta}(z,p)+\hat{\eta}(z,p)\hat{F}(z)\}, \tag{16}$$

where the non-adiabatic correction matrix $\hat{\eta}(z,p)$ is determined by the first order in the expansion in smallness of the recoil momentum of the density matrix

$$\hat{\rho}(z,p)=\hat{\sigma}(z,p)\,W(z,p)+\frac{\partial}{\partial p}\hat{\eta}(z,p)\,W(z,p)+\cdots \tag{17}$$

and satisfies the equation

$$\frac{p}{M}\frac{\partial}{\partial z}\hat{\eta}-\hat{L}^{(0)}\{\hat{\eta}\}=-\frac{1}{2}\left\{\widehat{\delta F}\,\hat{\sigma}+\hat{\sigma}\,\widehat{\delta F}\right\} \tag{18}$$

with the normalisation condition $\mathrm{Tr}\{\eta\}=0$. The operator $\delta F=\hat{F}(z)-F(z,p)$ is the force fluctuation operator for an atom in a light field.

Additionally, we note that in the process of reducing the QKE to the FP equation in the semiclassical expansion in the smallness of parameter of the photon momentum $\hbar k/\Delta p$ (12) uniquely determines the form of writing the diffusion contribution of the spontaneous diffusion coefficient (15) in the form $\partial^2/\partial p^2(D^{(\mathrm{s})}F)$. In this case, variability in the form of the choice of the diffusion contribution (3) or (4) arises only with respect to the induced diffusion $D^{(\mathrm{i})}$, the expression for which (16) is obtained by neglecting the smallness of the order of ε_{R}. As a result, due to the unambiguous notation of the diffusion contribution associated with the spontaneous diffusion coefficient, a comparative analysis is required for the FP equation in form (3)

$$\left(\frac{\partial}{\partial t}+\frac{p}{M}\frac{\partial}{\partial z}\right)W(z,p,t)=-\frac{\partial}{\partial p}F(z,p)\,W(z,p,t)$$
$$+\frac{\partial^2}{\partial p^2}(D^{(\mathrm{s})}(z,p)+D^{(\mathrm{i})}(z,p))\,W(z,p,t) \tag{19}$$

and in the form

$$\left(\frac{\partial}{\partial t}+\frac{p}{M}\frac{\partial}{\partial z}\right)W(z,p,t)=-\frac{\partial}{\partial p}F(z,p)\,W(z,p,t)$$
$$+\frac{\partial}{\partial p}D^{(\mathrm{i})}(z,p)\frac{\partial}{\partial p}W(z,p,t)$$
$$+\frac{\partial^2}{\partial p^2}D^{(\mathrm{s})}(z,p)\,W(z,p,t). \tag{20}$$

In contrast to quantum approaches [14–19], which make it possible to obtain exact solutions for the atomic density matrix in a light field, semiclassical approaches, despite the used approximations, are of particular interest, since the expressions for the kinetic coefficients of the FP equations allow one to qualitatively judge various features in atomic kinetics, as well as to describe and interpret different physical mechanisms of laser cooling [21, 22, 24, 25]. The stationary solution of the FP equation for atoms in a light field in form (3) or (4), neglecting the effects of localisation of atoms in the optical potential generated by the light field, can be obtained from expressions averaged over the spatial period for the force $\bar{F}(p)$ and the diffusion coefficients $\bar{D}(p)$:

$$W(p)=C\frac{\bar{D}(0)}{\bar{D}(p)}\exp\int_{-\infty}^{p}\bar{F}(p')/\left(\bar{D}(p')\,\mathrm{d}p'\right) \tag{21}$$

for the FP equation in form (3) and

$$W(p)=C'\exp\int_{-\infty}^{p}\bar{F}(p')/\left(\bar{D}(p')\,\mathrm{d}p'\right) \tag{22}$$

for an equation in form (4). Here C and C' are normalisation constants.

In our work, we analyse the limits of laser cooling by considering the numerical solution of FP equations (19) and (20), which will allow us to accurately take into account the effects of atomic localisation in the optical potential for atoms with different ε_{R} parameters. In addition, we note that the momentum distribution of atoms in a cooling laser field in some cases turns out to be substantially nonequilibrium [44] and therefore, strictly speaking, cannot be described in terms of temperature. In a number of cases, the shape of the resulting momentum distribution makes it possible to distinguish two fractions of atoms with different temperatures: a fraction of atoms with a narrow momentum distribution and having a sub-Doppler temperature, as well as a fraction with a significantly higher temperature, the Doppler one [35]. Therefore, in this paper, as a quantitative estimate of the laser cooling of the entire atomic ensemble, we present the results for the average kinetic energy of cold atoms.

3.1. Limits of laser cooling for a two-level model of an atom

The cooling of atoms in the field of a standing wave with uniform spatial polarisation can in some cases be reduced to the problem of laser cooling in the model of a two-level atom [45]. The two-level model for the problem of laser cooling was the primary approximation, and it has been studied quite well both in the framework of the semiclassical approach [21–23] and with full allowance for quantum recoil effects for atoms with different recoil parameters ε_R [46]. Here, we compared the data on the limits of laser cooling obtained on the basis of solutions of FP equations (19) and (20) taking into account the localisation effects and the numerical solution of QKE (1) based on the methods proposed by us in [18–20, 33] for atoms with different ε_R parameters (less than unity).

The results of the comparative analysis are shown in Fig. 1. First of all, we note that the results obtained on the basis of quantum and semiclassical approaches completely coincide for extremely small (10^{-3}) values of the parameter ε_R (Fig. 1a). Note that a comparative analysis without taking into account localisation effects based on solutions (21) and (22) leads to underestimated results for the semiclassical approach with respect to the quantum one for the kinetic energy of cold atoms with an increase in the intensity of the light field (parameter U_0) by ~20% at $U_0 = 300\hbar\omega_R$ (Fig. 2). This is due to the action of 'Sisyphus' mechanisms of laser cooling of atoms, which, at the minima of the optical potential (corresponding to the maxima of the light field intensity) at red field detunings $\delta < 0$, lead to local heating of the atoms [22, 29]. The limiting values of the kinetic energy in low-intensity fields correspond to the Doppler limit of laser cooling of atoms, achieved at the light field detuning $\delta = -\gamma/2$, which corresponds to the atomic temperature $k_B T_D = \hbar\gamma 7/20$ [22]. For an equilibrium distribution, this temperature corresponds to the average kinetic energy $E_{kin} = 7/(40\hbar\gamma)$, which is equivalent to $E_{kin} = 175\hbar\omega_R$ for atoms with $\varepsilon_R = 10^{-3}$ (Fig. 1a).

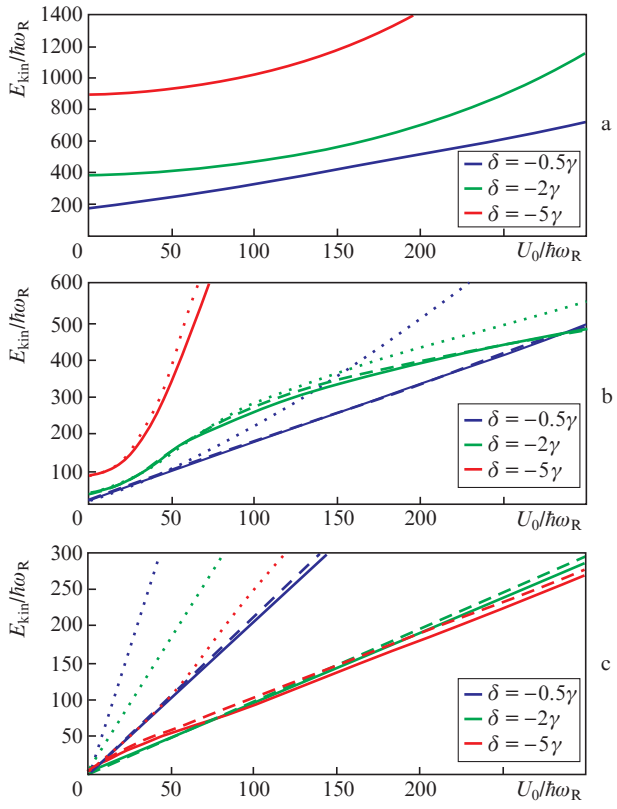

Figure 1. (Colour online) Kinetic energy of cold atoms in units of the recoil energy as a function of U_0 at various detunings of the light field for atoms with ε_R = (a) 10^{-3}, (b) 10^{-2} and (c) 10^{-1}, obtained within the two-level model of the atom (the light wave with uniform polarization). The solid curves show the results obtained based on the numerical solution of QKE (1), and the dotted and dashed curves show the results obtained in the framework of semiclassical approaches based on FP equations (19) and (20), respectively.

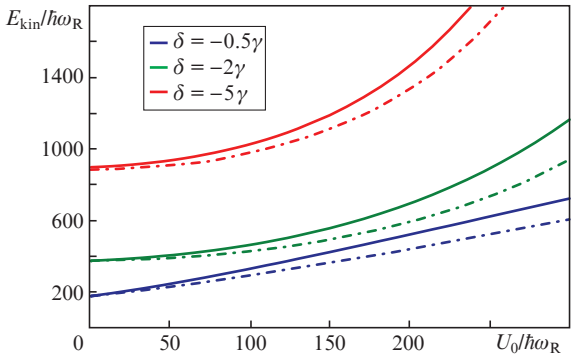

Figure 2. (Colour online) Kinetic energy of cold atoms in units of recoil energy as a function of U_0 at various detunings of the light field for atoms with $\varepsilon_R = 10^{-3}$, obtained within the two-level model of the atom. The solid curves show the results obtained on the basis of the numerical solution of QKE (1), and the dotted and dashed curves show the results obtained in the framework of semiclassical approaches based on FP equations (19) and (20) with expressions averaged over the spatial period for the force and diffusion, i.e. without taking into account the effects of atomic localisation in a light field.

For atoms with a large (10^{-2}) parameter ε_R (Fig. 1b), differences begin to appear in the results obtained on the basis of quantum and semiclassical approaches in form (19) and (20). We note that at low light field intensities, one can observe the similarity of the results obtained for atoms with $\varepsilon_R = 10^{-2}$ and 10^{-3}, which is a consequence of the 'scale law' noted by us in [46]. In this case, the results obtained using the semiclassical approach based on the FP equation in form (20) are in better agreement with the results of the numerical solution of QKE (1). This correspondence is also preserved for atoms with $\varepsilon_R = 10^{-1}$ (Fig. 1c).

3.2. Sub-Doppler laser cooling in the field with the $\sigma_+ - \sigma_-$ configuration

One of the most commonly used light field configurations with spatially nonuniform polarisation is the $\sigma_+ - \sigma_-$ configuration. In combination with a nonuniform magnetic field, it underlies the implementation of magneto-optical traps. For a comparative analysis, we consider the simplest model of an atom with levels $J_g = 1 \to J_e = 2$ degenerate in the angular momentum projection, which exhibit sub-Doppler laser cooling mechanisms [24].

The characteristic dependences of the average kinetic energy of cold atoms differ significantly from the results of the two-level model. Note that for atoms with extremely small ($\varepsilon_R \ll 1$) parameters, the average kinetic energy of atoms is below the Doppler limit, which is associated with the presence of sub-Doppler mechanisms of laser cooling, manifesting themselves in fields with a spatially uniform polarisation [24]. Nevertheless, even in the semiclassical limit (for atoms with

the extremely small parameter $\varepsilon_R = 10^{-3}$), there are differences in the results obtained on the basis of the quantum and semiclassical approaches. They manifest themselves most clearly at small detunings of the cooling field (Fig. 3a). These differences are associated with the exit from the resonance of the interaction of the field with atoms in the processes of absorption/emission of photons as a result of recoil effects, which is most pronounced for atoms with an insufficiently small ε_R (Fig. 3c). The influence of this effect decreases as the detuning $|\delta|$ is increased. In this case, the results of the semiclassical approach based on the FP equation in form (20) are in better agreement with the results of the numerical solution of QKE (1) even for atoms with an insufficiently small parameter ε_R.

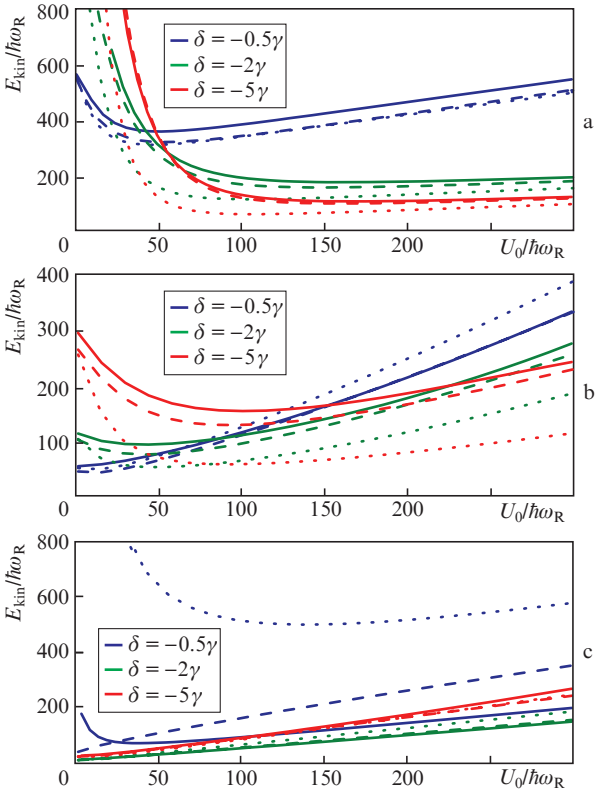

Figure 3. (Colour online) Kinetic energy of cold atoms in units of recoil energy as a function of U_0 at various detunings of the light field for atoms and ε_R = (a) 10^{-3}, (b) 10^{-2} and (c) 10^{-1} in the field with the $\sigma_+-\sigma_-$ configuration. The solid curves show the results obtained based on the numerical solution of QKE (1), and the dotted and dashed curves show the results obtained in the framework of semiclassical approaches based on the equations (19) and (20), respectively.

3.3. Sub-Doppler laser cooling in the field with the lin ⊥ lin configuration

When analysing the kinetics of atoms in the fields with the lin ⊥ lin configuration, we, as in Section 3.2, use the model of the optical transition $J_g = 1 \to J_e = 2$, which makes it possible to additionally compare the efficiency of laser cooling in these configurations of light fields. In the field with the lin ⊥ lin configuration formed by counterpropagating waves with orthogonal linear polarisations, deeper limits of laser cooling are reached than in the field with the $\sigma_+-\sigma_-$ configuration (Fig. 4).

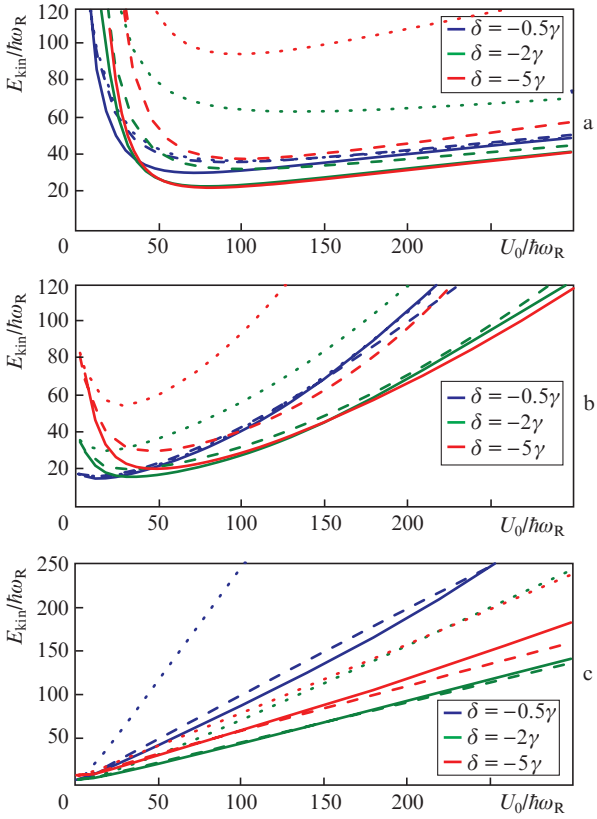

Figure 4. (Colour online) Kinetic energy of cold atoms in units of recoil energy as a function of U_0 at various detunings of the light field for atoms and ε_R = (a) 10^{-3}, (b) 10^{-2} and (c) 10^{-1} in the field with the lin ⊥ lin configuration. The solid curves show the results obtained based on the numerical solution of QKE (1), and the dotted and dashed curves show the results obtained in the framework of semiclassical approaches based on the equations (19) and (20), respectively.

Figure 4 shows the dependences of the average kinetic energy of cold atoms achieved during laser cooling in fields of different intensity (characterised by the parameter U_0) at different detunings for atoms with different parameters ε_R. We note that at red field detunings, $|\delta| > 1$, the expression for the induced diffusion coefficient at some low velocities of the atom locally takes negative values, which, for example, can be seen directly from the analytical expressions for the induced diffusion coefficient presented in [37]. In this case, the expression averaged over the spatial period for the induced diffusion coefficient $\overline{D^{(i)}}(p)$ (16) remains positive. This local behaviour of the diffusion coefficient looks unphysical and does not lead to a stationary numerical solution of FP equations (19) and (20), which indicates additional restrictions on the semiclassical expansion of the QKE that arise in fields with a spatially nonuniform polarisation, such as the lin ⊥ lin configuration. Therefore, for field detunings $|\delta| > 1$, FP equations (19) and (20) were solved numerically by using the induced diffusion coefficient averaged over the spatial period.

Based on the analysis of the results of laser cooling limits in the field with the lin ⊥ lin configuration (Fig. 4), it can be noted in the general case that the results obtained on the basis of the FP equation in form (20) are in better agreement with the results of the quantum approach obtained on the basis of numerical solution of QKE (1).

4. Conclusions

The semiclassical approach to solving the problem of the kinetics of atoms in light fields is one of the fundamental approaches used, among other things, for qualitative analysis and description of the mechanisms of laser cooling and trapping of atoms in light fields. Uncertainty in the description of the processes of fluctuations of forces acting on atoms in a light field and having a quantum nature from the point of view of the interaction of atoms with light, within the framework of the semiclassical approach, leads to the so-called Itô–Stratonovich dilemma, which is expressed in various forms of the Fokker–Planck equation, and also stochastic Langevin equations modelling the kinetics of atoms interacting with field photons.

In the framework of this work, we have performed a comparative analysis of the problem of laser cooling of atoms in light fields on the basis of a direct numerical solution of the quantum kinetic equation for the atomic density matrix (1), which makes it possible to accurately take into account quantum recoil effects in the interaction of atoms with field photons, as well as on the basis of a semiclassical approach in various forms of writing the FP equation. Formally, the conditions of the semiclassical approximation, in which equation (1) can be reduced to the FP equation for the distribution function of atoms in the phase space, is satisfied only for extremely small values of the parameter $\varepsilon_R \leqslant 10^{-3}$, where the difference in the form of FP equation (19) or (20) is insignificant. Indeed, in this limit, the solutions to Eqns (19) and (20) are in good agreement with each other and with the results obtained based on the numerical solution of QKE (1) with full allowance for quantum recoil effects. As the parameter ε_R increases, significant differences are observed in the solutions obtained on the basis of the FP equations in form (19) and (20). In this case, the results obtained on the basis of the semiclassical approach in the form of induced diffusion (20) are in better agreement with the results obtained on the basis of the numerical solution of QKE (1).

The result obtained in our work makes it possible to significantly expand the limits of applicability of the semiclassical approach up to $\varepsilon_R = 0.1$ for field detunings $|\delta| > \gamma/2$ when choosing the FP equation in form (20). This result is important because it makes it possible to justify the applicability of the semiclassical approach to laser cooling problems with an insufficiently small value of the parameter ε_R and to use in this case the FP equation for analysing the kinetics of atoms in light fields without resorting to resource-intensive calculations based on quantum approaches.

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